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Isometries of an affine euclidian space in finite dimension.
Application in dimensions 2 and 3.

Context: E affine euclidian space of dimension n.
$\vec{E}$ vectorial space associated.

I - Generalities

1.1. Definitions and first properties.

Def.1: an isometry (affine isometry) is a map $f: E \to E$ which preserves distances:
$\forall N, M \in E$, $\|f(M)f(N)\| = \|MN\|$.

$Is(E)$ is the set of isometries.

Thm. 2: a map $f: E \to E$ is an isometry if and only if it is affine and its linear part is an orthogonal map.

examples:
- translations (the linear part is the identity)
- orthogonal symmetries (see annexe 1)

1.2. Structure of $Is(E)$

prop 3. an isometry is bijective and $f \mapsto \vec{f}$ is a morphism

rmk $Is(E) < Aff(E)$

if $\vec{f} \in SO(E)$, f is a rigid motion. $Is^+(E)$ is the group of rigid motion.
if $\vec{f} \in O(E) \setminus SO(E)$, f is an indirect isometry. $f \in Is^-(E)$.

Thm 4: let f be in $Is(E)$. There is only one $\vec{u} \in \ker(\vec{f}-id)$ and one $g \in O(E)$ with one fixed point at least such that
$f = t_{\vec{u}} \circ g = g \circ t_{\vec{u}}$.

rmk: $\ker(\vec{f}-id)$ is the set of fixed points of $\vec{f}$. We denote $\ker(\vec{f}-id) = Inv(\vec{f})$.

Corollary 5: If $Inv(\vec{f}) = \{0\}$ then f has only one fixed point.

Corollary 6: $f \in Is(E)$ is a translation if and only if the quantity $\|M f(M)\|$ is constant, independant on $M \in E$.

Thm.7: let $(A_0, \dots A_n)$ be an affine frame and $(B_0 \dots B_n)$ a $(n+1)$-tuple such that $B_iB_j = A_iA_j$ for each i, j.
There is one and only one isometry f such that $f(A_i) = B_i$, $\forall i$.

II - Classification of isometries

2.1. Study of $O(E)$

motivation: $Is(E) \cong \vec{E} \rtimes O(\vec{E})$ and $Is^+(E) \cong \vec{E} \rtimes SO(\vec{E})$

Thm 8: let f be in $O(E)$. There is an orthonormal basis in which the matrix of f is

$$\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & R_{\theta_1} & \\ 0 & & & & \ddots \; R_{\theta_r} \end{pmatrix}$$

with $R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$.

Def. 9:
- a reflexion is a symmetry with respect to a plan (sub vectorial space of dimension 2).
- a reversal is a symmetry with respect to a sub-vectorial space of codimension 2.

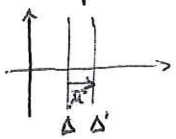
rmk: reflexions are in $O(E)$ and reversals in $SO(E)$.

Thm 10: (Cartan-Dieudonné)
each $f \in O(E)$ is the product of no more than n reflexions.

Corollary 11: each f in $SO(E)$ is the product of at most n reversals.

Corollary 12: • each f in $Is(E)$ is the product of at most $n+1$ reflexions.
• each f in $Is^+(E)$ is the product of at most $n+1$ reversal.

example: a translation is the product of 2 reflexions



$t_{\vec u} = s_{\Delta'} \circ s_\Delta$

Thm 13:
• $O(E)$ and $SO(E)$ are compact.
• $SO(E)$ is path connected.

Corollary 14: $Is^+(E)$ is path connected.

prop 15: $SO_3(\mathbb{R})$ is simple. $\boxed{\text{DEV 1}}$

Thm 16: (Polar decomposition)

$$\begin{array}{ccc} O_n(\mathbb{R}) \times S_n(\mathbb{R}) & \longrightarrow & GL_n(\mathbb{R}) \\ (O, S) & \longmapsto & OS \end{array}$$ is a homeomorphism.

Corollary 17: $\forall A \in GL_n(\mathbb{R})$, $\|A\|_2 = \sqrt{\rho({}^tAA)}$

Corollary 18: let G be a compact subgroup of $GL_n(\mathbb{R})$ such that $O_n(\mathbb{R}) < G$. Then $G = O_n(\mathbb{R})$.

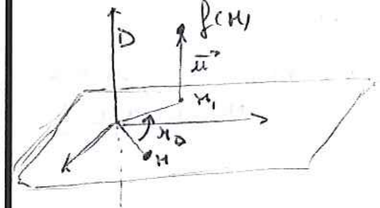
2.2. in dimension 1, 2 and 3.

rmk: $O_1(\mathbb{R}) \cong \{\pm 1\}$. So the only isometries are the translations and reflexions.

def 19: a glide symmetry the product of a translation and an orthogonal symmetry with the same direction
(Cf annexe 2)

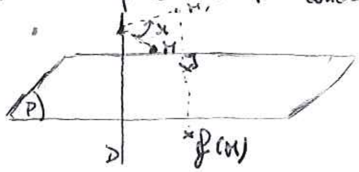
2D	translation	rotation	reflexion	glide symmetry
invariant point	$\emptyset$	1 point	1 line	$\emptyset$
invariant direction	1 direction	$\emptyset$	1 direction	1 line
decomposition in reflexion	2 lines //	2 secante lines	1 line	3 lines

def 20: a screw-displacement is a product $t_{\vec u} \circ r_D$ where r_D is a rotation of axis D and $\vec u \in \vec D$.



prop 21: a rigid motion of the space is a screw-displacement.

def 22: an improper rotation is a product $r_D \circ s_P$ where r_D is a rotation of axis D, s_P the reflexion with respect to P and $P = D^\perp$.



3D	
Inv(f)	f
space	identity
plane	reflexion
line	rotation ($\neq$ id)
point	improper rotation
$\emptyset$	translation, glide reflexion, screw-displacement

III - Shape-preserving isometries

3. 1. pattern in the plane

def.23: let P be a compact convexe set of the plan so that $\mathring{P} \neq \phi$. A *wallpaper group* is a subgroup of $Is^+(E)$ such that:

- $\bigcup_{g \in G} g(P) = E$
- $(g(\mathring{P}) \cap h(\mathring{P}) \neq \phi) \Longrightarrow (g(P) = h(P))$

Thm 24: There is only five wallpaper groups.

(Cf annexe 3) $\boxed{\text{DEV. 2}}$

rmk:
- if we allow $G < Is(E)$, then there is 17 groups.
- These groups are used in crystallography.

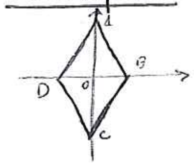
3. 2. preserving a part

def 25: $P \subset E$, the space. $Is(P)$ (resp. $Is^{\pm}(P)$) is the sub-set of $Is(E)$ (resp. $Is^{\pm}(E)$) such that: $\forall f \in Is(P)$, $f(P) = P$.

Thm 26:
- $Is(P)$ is a sub-group of $Is(E)$
- $Is^+(P)$ is a sub-group of $Is(P)$.
- if $s \in Is^-(P)$, $Is^+(P) \cong s\, Is^-(P)$

Thm 27: if $P = \{A_0 \dots A_{m-1}\}$ and $O =$ isobary(P) then: $\forall f \in Is(P)$ $f(O) = O$.

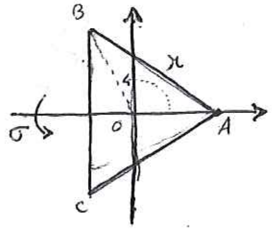
example: P: a rombus, non-square, in the plan.



$Is(P) = \{id, s_O, s_{AC}, s_{BD}\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

Thm 28: If P is a regular polygon with m vertices $(m \geq 3)$ then $Is(P) \cong D_m$.

example:



$Is(ABC) = \langle r, \sigma \rangle \cong D_3$

$\boxed{\text{DEV. 3}}$

Thm 29: The finite sub-groups of $SO_3(\mathbb{R})$ are
- $\mathbb{Z}/m\mathbb{Z}$
- D_m
- $\mathcal{A}_4$
- $\mathfrak{S}_4$
- $\mathcal{A}_5$
- $\{id\}$ $(m \geq 2)$

prop.30:

$Is^+(\text{tetrahedron}) \cong \mathcal{A}_4$

$Is^+(\text{cube}) \cong \mathfrak{S}_4$

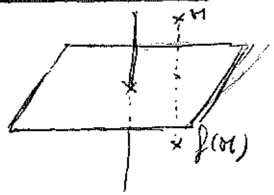
$Is^+(\text{octahedron}) \cong \mathfrak{S}_4$

$Is^+(\text{dodecahedron}) \cong \mathcal{A}_5$

$Is^+(\text{isocahedron}) \cong \mathcal{A}_5$

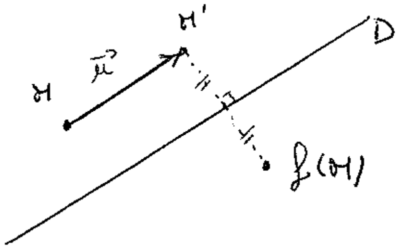
rmk: the finite sub-groups of $PGL_2(\mathbb{C})$ are the same.

Annexe 1:



orthogonal symmetry with respect to a plan

Annexe 2.



glide symmetry.

Annexe 3: wallpaper groups