Control theory and splitting methods

Karine Beauchard[*], Adrien Laurent[*], Frédéric Marbach[†]

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Abstract

Our goal is to highlight some of the deep links between numerical splitting methods and control theory. We consider evolution equations of the form $\dot{x} = f_0(x) + f_1(x)$, where f_0 encodes a non-reversible dynamic, so that one is interested in schemes only involving forward flows of f_0. In this context, a splitting method can be interpreted as a trajectory of the control-affine system $\dot{x}(t) = f_0(x(t)) + u(t)f_1(x(t))$, associated with a control u which is a finite sum of Dirac masses. The general goal is then to find a control such that the flow of $f_0 + u(t)f_1$ is as close as possible to the flow of $f_0 + f_1$.

Using this interpretation and classical tools from control theory, we revisit well-known results concerning numerical splitting methods, and we prove a handful of new ones, with an emphasis on splittings with additional positivity conditions on the coefficients. First, we show that there exist numerical schemes of any arbitrary order involving only forward flows of f_0 if one allows complex coefficients for the flows of f_1. Equivalently, for complex-valued controls, we prove that the Lie algebra rank condition is equivalent to the small-time local controllability of a system. Second, for real-valued coefficients, we show that the well-known order restrictions are linked with so-called "bad" Lie brackets from control theory, which are known to yield obstructions to small-time local controllability. We use our recent basis of the free Lie algebra to precisely identify the conditions under which high-order methods exist.

Contents

[*]Univ Rennes, ENS Rennes, INRIA, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France

[†]DMA, École normale supérieure, Université PSL, CNRS, 75005 Paris, France

1 Introduction

In this article, we highlight the deep links between numerical splitting methods and control theory and use this comparison to provide new proofs of known results and conjectures of the order theory of splitting methods. We focus on situations where f_0 encodes a heuristically non-reversible dynamic, so that one is interested in schemes only involving forward flows of f_0. An overview of the main results is presented in Section 1.3.

1.1 Order theory for splitting methods

Splitting methods Splitting methods aim at solving numerically evolutionary problems of the general form

$$\dot{x}(t) = f_0(x(t)) + f_1(x(t)) + \cdots + f_p(x(t)), \tag{1.1}$$

where the flows associated to the vector fields f_n are easy to integrate numerically with high precision or an exact solution is available. We consider $p = 1$ for simplicity in this paper. For a given time T, a splitting method approximates $x(T)$ by composing flows associated to the f_i. A standard splitting method is of order N for solving (1.1) if for all smooth vector fields f_0, f_1, the following estimate holds

$$x(T) = e^{\alpha_1 T f_0} e^{\beta_1 T f_1} \ldots e^{\alpha_k T f_0} e^{\beta_k T f_1} x(0) + \underset{T\to 0}{O}(T^{N+1}). \tag{1.2}$$

For instance, the Lie–Trotter splitting is of order one:

$$x(T) = e^{T f_0} e^{T f_1} x(0) + \underset{T\to 0}{O}(T^2), \tag{1.3}$$

and the Strang splitting is of order two:

$$x(T) = e^{T/2 f_0} e^{T f_1} e^{T/2 f_0} x(0) + \underset{T\to 0}{O}(T^3). \tag{1.4}$$

Splitting methods are widely popular methods used for their straightforward implementation, versatility, accuracy, and stability. They also have good geometric behaviour [45, 31, 11] for preserving, for instance, energy, volume, symmetries, or symplecticity. They are heavily used for the approximation of ODEs and PDEs (see, for instance, [8, 42, 27]), and also in the stochastic setting, for instance in molecular dynamics [39, 17], with the popular BAOAB or UBU schemes for kinetic Langevin (see also [2]).

Order theory In this paper, we are interested only in the creation of high-order splittings and we ignore the stability analysis, as well as the preservation of geometric properties. The order theory of splitting methods relies on the Baker–Campbell–Hausdorff formula or Magnus formula [44] and it is possible to create methods of any high order, with for instance composition methods. The derivation of the order conditions is found for instance in [45] and the modern formulation uses the algebraic framework of free Lie algebras [52] and Hopf algebras (see, for instance, the word series [48, 49] and the review [11, Sec. 2]). The formalism of free Lie algebras has a long successful history in control theory [58, 5], and we extend the approach here to the study of splitting methods. An important feature of the analysis is the choice of basis of the free Lie algebras. In particular, we will use the basis recently introduced in [7]. The analysis also shares similarities with Lie group methods [36]. Indeed, the study of Lie group methods relies on the study of the compositions of frozen and exact flow exponentials, and the latter is also the object of interest in the context of splitting methods (see [47, 43, 46, 26, 29, 25, 1]). The analysis of splitting methods relies on words, while the analysis of Runge–Kutta like integrators relies on trees. In this context, the analysis of splitting methods is simpler and more compact. We cite in particular the related algebraic tree formalisms [15, 31] of Butcher trees, [35, 41] for exponential integrators, [14, 13] for splitting schemes with low regularity initial data, and [53, 38, 12] for stochastic integrators.

Order theory for semigroups The order theory of splitting methods becomes more involved when additional positivity conditions are added. We study in particular the conditions of existence of splitting schemes of high order with coefficients in $(\mathbb{A}, \mathbb{B})$, that is, when $\alpha_i \in \mathbb{A}$, $\beta_i \in \mathbb{B}$ for all i. For non-reversible problems (heat equation, stochastic differential equations,...), a condition of the form $\alpha_i > 0$ on the coefficients in (1.2) must be imposed. In this context, it is known [55, 59, 30, 8] that the maximum order for splitting methods with coefficients in $(\mathbb{R}^+, \mathbb{R})$ and $(\mathbb{R}^+, \mathbb{R}^+)$ (sometimes called forward splitting methods) is $N = 2$. A possible solution is the use of splitting methods with complex coefficients. It first appeared in the context of Hamiltonian systems [18] and quantum mechanics [3, 50] with low order. It then appeared simultaneously in the works [16, 34] for parabolic problems, in the spirit of [33]. The large number of complex solutions for the order conditions offers more flexibility in the choice of coefficients, and can lead to schemes with smaller truncation errors and new symmetries. On the other hand, the use of complex arithmetic introduces an additional cost for solving real problems, and extending the flows to the complex plane has to be done carefully in order to avoid order reductions. The papers [16, 34] use symmetric composition methods to create splitting methods in $(\mathbb{C}^+, \mathbb{C}^+)$ up to order 14 (i.e. where all coefficients are complex numbers with positive real part), though the error constants deteriorate in some cases [9]. In [9], it is proven that splitting methods in $(\mathbb{C}^+, \mathbb{C}^+)$ exist up to order 44, by building upon a splitting method of order 6 in $(\mathbb{R}^+, \mathbb{C}^+)$. In this paper, we prove in particular that splittings in $(\mathbb{R}^+, \mathbb{C})$ exist up to any order, giving a positive answer to the open question of [9, Remark 2.7] (with unconstrained complex β_i).

Commutator flows and degeneracies An alternative solution to go beyond the order barrier is to introduce flows associated to specific commutators in the splitting methods, assuming that these flows are explicitly available. The first such method in the literature is the Takahashi–Imada splitting [60] (see also [54, 37, 22]). In the context of Hamiltonian dynamics, such splittings with commutators appear under the name *splitting methods with modified potentials* [40, 54, 62]. We refer to [11, Sections 3.2 and 8] for an extensive list of the use of splittings with commutators in the literature. It is important to mention that the splitting schemes using commutators are of two different types in the literature. The first type concerns splittings with commutators, that are, general splitting methods where one allows the use of commutator flows. The second type is tied to specific systems that satisfy degeneracies: the commutators satisfy some identities for the specific f_i considered and the splitting schemes use these degeneracies. Following the open questions in [8], we provide new existence results of splitting schemes in $(\mathbb{R}^+, \mathbb{R})$ with commutators and necessary degeneracy conditions to obtain high order splitting schemes.

Link with control theory The link between splitting methods with $\mathbb{A} = \mathbb{R}^+$ and control theory is the following: a splitting method can be seen as a trajectory of the control system

$$\dot{x}(t) = f_0(x(t)) + u(t) f_1(x(t)) \tag{1.5}$$

associated with a control u which is a sum of Dirac masses (α_j is the duration of the step and β_j the amplitude of the jump) such that the associated flow of the time-varying vector field $f_0 + u(t) f_1$ defined on $[0, T]$ is approximately $e^{T(f_0+f_1)}$ for some $T > 0$.

With this in mind, we revisit known results on numerical splitting methods, and prove new ones. In particular, we identify Lie brackets that are obstructions to both high order numerical splitting methods and small-time local controllability.

1.2 Definitions and notations

1.2.1 Formal brackets and evaluated Lie brackets

Definition 1.1 (Formal brackets)**.** *Let $X = \{X_0, X_1, \dots, X_m\}$ for $m \geq 1$ be a finite set of non-commutative indeterminates. We denote by $\mathrm{Br}(X)$ the* free magma *over X, which can be defined by induction: $X \subset \mathrm{Br}(X)$ and, if $a, b \in \mathrm{Br}(X)$, then the ordered pair (a, b) belongs to $\mathrm{Br}(X)$. For $b \in \mathrm{Br}(X)$, we denote by $|b|$ its length and by $n_j(b)$ the number of occurrences of X_j in b. For $a, b \in \mathrm{Br}(X)$, we define $\mathrm{ad}_a(b) := (a, b)$ and by induction $\mathrm{ad}_a^{n+1}(b) := (a, \mathrm{ad}_a^n(b))$. For instance (X_0, X_0) and $b' := \mathrm{ad}_{X_1}^2(X_0) = (X_1, (X_1, X_0))$ are elements of $\mathrm{Br}(X)$ respectively of lengths 2 and 3, and $n_1(b') = 2$, $n_0(b') = 1$.*

Definition 1.2. *We give explicit names to some elements of $\mathrm{Br}(X)$:*

$$M_0 := X_1 \text{ and } M_{\nu+1} := (M_\nu, X_0) \text{ for every } \nu \in \mathbb{N}, \tag{1.6}$$

$$W_j := \mathrm{ad}_{M_{j-1}}^2(X_0) = (M_{j-1}, M_j) \text{ for every } j \in \mathbb{N}. \tag{1.7}$$

These notations will prove useful in stating our main results. In particular, we have $M_1 = (X_1, X_0)$ and $W_1 = (X_1, (X_1, X_0))$.

Definition 1.3 (Lie bracket of vector fields)**.** *For smooth vector fields f, g, we denote their usual Lie bracket by $[f, g] =: \mathrm{ad}_f(g)$ and use the adjoint representation $\mathrm{ad}_f^{n+1}(g) := [f, \mathrm{ad}_f^n(g)]$ for $n \in \mathbb{N}$.*

Definition 1.4 (Evaluated Lie bracket)**.** *If $f_0, \dots, f_m$ are smooth vector fields and $b \in \mathrm{Br}(X)$, then f_b denotes the vector field obtained by replacing the indeterminates X_j with the corresponding vector fields f_j in the iterated bracket b.*

We find for instance $f_{(X_1,(X_2,X_3))} = [f_1, [f_2, f_3]]$ and $f_{W_1} = \mathrm{ad}_{f_1}^2(f_0)$.

1.2.2 Splitting methods

In the sequel, we rely on the following definition of order of a splitting method.

Definition 1.5 (Order of a splitting method with $(\mathbb{A}, \mathbb{B})$ coefficients)**.** *Let $\mathbb{A}, \mathbb{B} \subset \mathbb{C}$ and $N \in \mathbb{N}^*$. We say that $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{A}^k$ and $\beta = (\beta_1, \dots, \beta_k) \in \mathbb{B}^k$ for some $k \in \mathbb{N}^*$ is a* splitting method of order (at least) N *when, for every smooth vector fields f_0, f_1 on $\mathbb{K}^d$ (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $d \in \mathbb{N}^*$),*

$$e^{T(f_0+f_1)} = e^{\alpha_1 T f_0} e^{\beta_1 T f_1} \cdots e^{\alpha_k T f_0} e^{\beta_k T f_1} + \underset{T \to 0}{O}(T^{N+1}) \tag{1.8}$$

in the following sense, used throughout the article: for every $x_0 \in \mathbb{K}^d$, there exist $C = C(f_0, f_1, x_0)$ such that, for every $T > 0$,

$$|e^{T(f_0+f_1)} x_0 - e^{\alpha_1 T f_0} e^{\beta_1 T f_1} \cdots e^{\alpha_k T f_0} e^{\beta_k T f_1} x_0| \leq C T^{N+1}. \tag{1.9}$$

In Definition 1.5 and throughout this paper, the role of the coefficients $\alpha_i \in \mathbb{A}$ (associated with flows of f_0) and $\beta_i \in \mathbb{B}$ (associated with flows of f_1) is not symmetric, since we will most often consider situations where only forward flows of f_0 can be used, while there will be no such constraint on flows of f_1.

In Definition 1.5, the parameters k, α, β must be independent of f_0, f_1 and T. A weaker notion allows the parameters k, α, β to depend on a set $\mathcal{F}$ of vector fields, which might have some degeneracies, potentially allowing for high-order methods.

Definition 1.6 (Splitting method relative to vector fields)**.** *Let $\mathcal{F}$ be a set of (pairs of) vector fields. We say that a splitting method is* of order N relative to $\mathcal{F}$ *when* (1.8) *holds for any $(f_0, f_1) \in \mathcal{F}$.*

Splitting methods of Definition 1.5 only involve flows of f_0 and f_1. For some systems, although the flow of $f_0 + f_1$ is not directly available, one might have access to the flows of some specific commutators of f_0 and f_1 (for example, the flow of $f_{W_1} = [f_1, [f_1, f_0]]$, or more generally of some f_b for $b \in \mathrm{Br}(X)$). This leads to the following definition.

Definition 1.7 (Splitting method involving commutator flows)**.** *Let $X = \{X_0, X_1\}$, $m \in \mathbb{N}^*$ and $b_1 := X_1, b_2, \ldots, b_m \in \mathrm{Br}(X)$. With the notations of Definition 1.5, a* splitting method of order N involving X_0 and $b_1, b_2, \ldots, b_m$ *is given by the additional data of $c = (c_1, \ldots, c_k) \in \{b_1, \ldots, b_m\}^k$ such that*

$$e^{T(f_0+f_1)} = e^{\alpha_1 T f_0} e^{\beta_1 T^{|c_1|} f_{c_1}} e^{\alpha_2 T f_0} e^{\beta_2 T^{|c_2|} f_{c_2}} \cdots e^{\alpha_k T f_0} e^{\beta_k T^{|c_k|} f_{c_k}} + \underset{T \to 0}{O}(T^{N+1}). \tag{1.10}$$

We impose without loss of generality that between flows of the form $e^{\beta f_c}$, there always is a term $e^{\alpha f_0}$ with $\alpha > 0$.

1.2.3 Controllability

Given smooth vector fields $f_0, f_1, \ldots, f_m$ defined on a neighborhood of $0 \in \mathbb{K}^d$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we consider the control-affine system

$$\dot{x}(t) = f_0(x(t)) + u_1(t) f_1(x(t)) + \cdots + u_m(t) f_m(x(t)), \tag{1.11}$$

where $u_i \in L^1(0, T)$ are the controls. When there is no control in front of $f_0 \neq 0$, such a system is called “with drift” and one typically assumes that $f_0(0) = 0$ and studies its behavior near the equilibrium $(x, u) = (0, 0)$. When there is a control $u_0(t)$ in front of f_0 (or equivalently when $f_0 = 0$), the system is called “driftless” and $(x, u) = (0, 0)$ is still an equilibrium. A handful of concepts of controllability exist (see [7, Section 1.2] for further discussion). For the sake of clarity, we will use the following definition in both cases.

Definition 1.8 (Small-state Small-Time-Local-Controllability)**.** *We say that* (1.11) *is* small-state-STLC *when, for every $T > 0$, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every target state $x^* \in B(0, \delta)$, there exists $u \in L^1((0, T); \mathbb{K}^m)$ such that the solution to* (1.11) *associated with u and initial data $x(0) = 0$ satisfies $x(T) = x^*$ and $x([0, T]) \subset B(0, \varepsilon)$.*

1.3 Main results

1.3.1 Arbitrary order $(\mathbb{R}, \mathbb{R})$ splitting methods

It is well-known that splitting methods with $(\mathbb{R}, \mathbb{R})$ coefficients of arbitrary order exist. With our notations, one has the following classical result.

Theorem 1.9. *For every $N \in \mathbb{N}^*$, there exists an $(\mathbb{R}, \mathbb{R})$ splitting method of order N with a number of flows at most $2 \dim(\mathcal{L}^N(X)) - 1$ (which is bounded above[1] by 2^{N+1}).*

Without any constraint on the real coefficients, an $(\mathbb{R}, \mathbb{R})$ splitting method corresponds to a trajectory of the driftless control-affine system

$$\dot{x}(t) = u_0(t) f_0(x(t)) + u_1(t) f_1(x(t)) \tag{1.12}$$

with state $x(t) \in \mathbb{R}^d$ and control $(u_0, u_1) : \mathbb{R}^+ \to \mathbb{R}$, where u_0 and u_1 are piecewise constant functions with disjoint supports, and do not depend on (f_0, f_1).

In control theory, the analogue of Theorem 1.9 is the following result, known as the Chow–Rashevskii necessary and sufficient condition for the controllability of driftless control-affine systems [23, 51], leading to the so-called “Lie algebra rank condition” $\mathrm{Lie}_{\mathbb{R}}(f_0, f_1)(0) = \mathbb{R}^d$, where $\mathrm{Lie}(f_0, f_1)$ denotes the Lie algebra spanned on $\mathbb{R}$ by f_0 and f_1.

[1] See Section 2.1 for a definition of $\mathcal{L}^N(X)$. By Witt's formula [63], $\dim \mathcal{L}^N(X) = \frac{1}{N} \sum_{d|N} \mu(d) |X|^{N/d}$, where μ is the Möbius function.

Theorem 1.10. *Let f_0, f_1 be real-analytic vector fields on a neighborhood of* 0 *in* $\mathbb{R}^d$. *System* (1.12) *is small-state-STLC with real-valued controls u_0, u_1 if and only if* $\mathrm{Lie}_{\mathbb{R}}(f_0, f_1)(0) = \mathbb{R}^d$.

To underline the similarity between Theorems 1.9 and 1.10, we will give a proof of both relying on the same underlying abstract result Proposition 5.5.

1.3.2 Arbitrary order $(\mathbb{R}^+, \mathbb{C})$ splitting methods

In [9, Remark 2.7], it is mentioned that the existence of splitting methods with $(\mathbb{R}^+, \mathbb{C})$ coefficients is an open question. We provide here the following positive answer.

Theorem 1.11. *For every $N \in \mathbb{N}^*$, there exists an $(\mathbb{R}^+, \mathbb{C})$ splitting method of order N with a number of flows at most* $N^N \dim(\mathcal{L}^N(X))$.

Theorem 1.11 is therefore an arbitrary order extension of the lower order methods recalled in Section 1.1. It is however an abstract existence result, and the proof that we give is not constructive.

An $(\mathbb{R}^+, \mathbb{C})$ splitting method corresponds to a trajectory of the scalar-input control-affine system (1.5) with state $x(t) \in \mathbb{C}^d$ and control $u : \mathbb{R}^+ \to \mathbb{C}$, where u is a finite sum of Dirac masses with complex amplitudes. The following control statement is the analogue of Theorem 1.11 for the control system (1.5).

Theorem 1.12. *Let f_0, f_1 be holomorphic vector fields on a neighborhood of* 0 *in $\mathbb{C}^d$ with* $f_0(0) = 0$. *System* (1.5) *is small-state-STLC with complex-valued control u if and only if* $\mathbb{C}^d = \mathrm{Lie}_{\mathbb{C}}(f_0, f_1)(0)$.

For control theorists, Theorem 1.12 can be unsettling at first sight. Indeed, for control-affine systems with drift of the form (1.5), no necessary and sufficient condition for controllability is known, when one considers real-valued controls. In particular, the Lie algebra rank condition does not imply the controllability as can be checked on the epitomal example on $\mathbb{R}^2$ given by

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1^2 \end{cases} \tag{1.13}$$

for which $f_1(0) = (1, 0)$ and $[f_1, [f_1, f_0]](0) = (0, 2)$, so that $\mathrm{Lie}(f_0, f_1)(0) = \mathbb{R}^2$ but, for real-valued u, $x_1 \in \mathbb{R}$ so $\dot{x}_2 \geq 0$, preventing controllability, since one cannot reach a target state with $x_2^* < 0$ starting from the initial state $(0, 0)$.

Theorem 1.12 is nevertheless reasonable, since, as in (1.13), all known obstructions to controllability rely on the presence of positive drifts in the dynamics (see [7] for an extended explanation). In particular, one easily checks that (1.13) is indeed controllable with complex-valued controls using that $i^2 = -1$.

Theorem 1.12 can be seen as a consequence of Sussmann's $S(\theta)$ sufficient condition for STLC of [58] (see Proposition 8.2 below). In this article, we prefer to adapt Sussmann's proof to this complex-valued context to emphasize that it becomes simpler and that no compensation needs to be done (unlike in Proposition 8.2). Moreover, to underline the similarity between Theorems 1.11 and 1.12, we will give a proof of both relying on the same underlying abstract result Proposition 6.4.

1.3.3 The first obstruction to controllability and $(\mathbb{R}^+, \mathbb{R})$ splitting methods

It is well-known that splitting methods with $(\mathbb{R}^+, \mathbb{R})$ coefficients suffer from severe order limitations. In particular, one has the following result (see e.g. [8]).

Theorem 1.13. *The maximal order of an $(\mathbb{R}^+, \mathbb{R})$ splitting method is 2.*

We claim in the sequel that the cause of this order restriction is the positive-definiteness of the coordinate associated with the "bad" bracket $W_1 = (X_1, (X_1, X_0))$ for the free system (3.1), and that this is linked with well-known obstructions to controllability.

An $(\mathbb{R}^+, \mathbb{R})$ splitting method corresponds to a trajectory of the control system (1.5) with state $x(t) \in \mathbb{R}^d$ and control $u : \mathbb{R}^+ \to \mathbb{R}$, where u is a finite sum of Dirac masses with real amplitudes. In control theory, the following result, due to Sussmann [57, Proposition 6.3] (see also [7, Theorem 1.10] for a modern proof using the Magnus formula and denying small-state-STLC) is the first necessary condition for controllability.

Theorem 1.14. *Let f_0, f_1 be smooth vector fields on a neighborhood of $0 \in \mathbb{R}^d$ such that $f_0(0) = 0$. If the system* (1.5) *is small-state-STLC then $f_{W_1}(0) \in \operatorname{span}\{f_{M_\nu}(0); \nu \in \mathbb{N}\}$.*

An analog of Theorem 1.14 for splitting methods is the following result.

Theorem 1.15. *Let f_0, f_1 be smooth vector fields on $\mathbb{R}^d$. If there exists an $(\mathbb{R}^+, \mathbb{R})$ splitting method of order 3 relative to (f_0, f_1), then f_{W_1} and f_{M_2} are linearly dependent.*

The slight difference in the compensation conditions between Theorem 1.14 and Theorem 1.15 can be explained in the following way. On the one hand, small-state-STLC is concerned with a regime where the control u is small, and the amplitude of the movement of the state in the direction of $f_{W_1}(0)$ is quadratic in u, whereas the movements along the $f_{M_\nu}(0)$ are linear in u, so that all such terms are equally capable of absorbing the drift associated with W_1. On the other hand, in Definition 1.5, a splitting method uses coefficients which are independent of the time T, hence compensations must occur between brackets of the same total length, and the only other bracket of length 3 is $M_2 = ((X_1, X_0), X_0)$. If the coefficients were allowed to depend in a polynomial way on T, the condition could involve f_{W_1} and $\{f_{M_\nu}; \nu \leq 2\}$.

Another way to apprehend the fact that W_1 is indeed the root cause of Theorem 1.13 is the following result, which states that, if one is able to compute the flow of $\pm f_{W_1}$ (in fact, one could prove that the flow of $-f_{W_1}$ is sufficient), then one can overcome the order restriction. A splitting method involving X_0 and X_1, W_1 corresponds to a trajectory of the control system

$$\dot{x}(t) = f_0(x(t)) + u_1(t) f_1(x(t)) + u_2(t) f_{W_1}(x(t)) \tag{1.14}$$

with state $x(t) \in \mathbb{R}^d$ and control $u = (u_1, u_2) : \mathbb{R}^+ \to \mathbb{R}^2$, where u_1, u_2 are finite sums of Dirac masses with disjoint supports and real amplitudes.

Theorem 1.16. *There exists an $(\mathbb{R}^+, \mathbb{R})$ splitting method of order 4 involving X_0 and X_1, W_1.*

Such an observation was already made in [8, Section 5], in which an $(\mathbb{R}^+, \mathbb{R}^+)$ splitting method, involving X_0 and $X_1, -W_1$ is constructed. We give here a proof relying on control theory, which only gives an $(\mathbb{R}^+, \mathbb{R})$ scheme, but can easily be adapted to prove the existence of higher order schemes (see below). As in Theorem 1.11 above and Theorem 1.19 below, all our positive results are non-constructive abstract existence results.

1.3.4 The next obstructions to controllability and splitting

The first obstruction W_1 is far from being the only one. In fact, even in situations where the first necessary conditions $f_{W_1}(0) \in \operatorname{span}\{f_{M_\nu}(0); \nu \in \mathbb{N}\}$ (for control theory) or f_{W_1} and f_{M_2} are linearly dependent (for splitting) hold, or one incorporates the flow of f_{W_1}, other obstructions occur. For control theory, the first and third authors have started a classification of obstructions in [7]. We refer the interested reader to this reference, and we will omit statements about controllability from now on, although, as above, one could continue to exhibit a very direct correspondance between both domains.

The next obstruction is caused by $W_2 = \operatorname{ad}^2_{(X_1, X_0)}(X_0)$ (a bracket of length 5, with 2 times X_1 and 3 times X_0). We will prove the following statements.

Theorem 1.17. *Let f_0, f_1 be smooth vector fields on $\mathbb{R}^d$ such that $f_{W_1} = 0$. If there exists an $(\mathbb{R}^*, \mathbb{R})$ splitting method of order 5 relative to (f_0, f_1), then f_{W_2} and f_{M_4} are linearly dependent.*

Theorem 1.18. *The maximal order of an* $(\mathbb{R}^+, \mathbb{R})$ *splitting method involving* X_0 *and* X_1, W_1 *is 4.*

Theorem 1.19. *There exists an* $(\mathbb{R}^+, \mathbb{R})$ *splitting method involving* X_0 *and* X_1, W_1, W_2 *of order 6.*

Theorem 1.18 yields a theoretical proof of the numerical observations of [8, Section 5].

As can be expected, the game goes on, and further obstructions exist at higher order. We refer to Sections 7 and 8 for further results. In particular, our approach allows to prove that all the W_j for $j \in \mathbb{N}^*$ of (1.7) yield obstructions to splitting.

1.4 Structure of the article

The first sections introduce the appropriate prerequisites, which might already be well known or folklore knowledge for readers accustomed to the field. In Section 2 we give prerequisites about algebraic tools: free (Lie) algebras, Lie groups, Hall bases. In Section 3, we introduce formal differential equations and some of their properties. In Section 4, we recall error estimates for the Magnus expansion of solutions to ordinary nonlinear differential equations.

We then move on to the proofs of the main results. In Section 5, we investigate the $(\mathbb{R}, \mathbb{R})$ case and prove Theorem 1.9 and Theorem 1.10. In Section 6, we investigate the $(\mathbb{R}^+, \mathbb{C})$ case and prove Theorem 1.11 and Theorem 1.12. In Section 7, we prove the statements concerning the maximal possible order of $(\mathbb{R}^+, \mathbb{R})$ methods. In Section 8, we prove the statements concerning the existence of methods achieving the maximal possible order of $(\mathbb{R}^+, \mathbb{R})$ methods using commutator flows. Eventually, in Section 9, we prove the statements concerning the degeneracies implied by the existence of methods with an order exceeding the maximal possible order.

2 Algebraic tools

In Section 2.1, we introduce several free algebras, that allow to define in Section 2.2 a Lie group, and to study the formal differential equation in Section 3. Finally, in Section 2.3, we recall the definition and an example of a Hall basis of $\mathcal{L}(X)$.

2.1 Free algebras

Let $X = \{X_0, \dots, X_m\}$ be a finite set of non commutative indeterminates and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, which will serve as a base field for all vector spaces and algebras.

Definition 2.1 (Free algebra)**.** *$\mathcal{A}(X)$ denotes the* free associative algebra *generated by X over the field $\mathbb{K}$, i.e. the unital associative algebra of polynomials of the non commutative indeterminates $X_0, \dots, X_m$ with coefficients in $\mathbb{K}$. $\mathcal{A}(X)$ can be seen as a graded algebra:*

$$\mathcal{A}(X) = \bigoplus_{n \in \mathbb{N}} \mathcal{A}_n(X), \tag{2.1}$$

where $\mathcal{A}_n(X)$ is the finite-dimensional $\mathbb{K}$-vector space spanned by monomials of degree n over X. In particular $\mathcal{A}_0(X) = \mathbb{K}$ and $\mathcal{A}_1(X) = \mathrm{span}_{\mathbb{K}}(X)$.

$\mathcal{A}(X)$ is endowed with a natural structure of Lie algebra, the Lie bracket operation being defined by $[a, b] := ab - ba$. This operation satisfies $[a, a] = 0$ and the Jacobi identity

$$[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0. \tag{2.2}$$

Definition 2.2 (Free Lie algebra)**.** *$\mathcal{L}(X)$ denotes the* free Lie algebra *generated by X over the field $\mathbb{K}$, which is defined as the Lie subalgebra generated by X in $\mathcal{A}(X)$. It can be seen as the smallest linear subspace of $\mathcal{A}(X)$ containing all elements of X and stable by the Lie bracket.*

There is a natural evaluation *mapping* E *from the free magma* $\mathrm{Br}(X)$ *defined in Definition 1.1 to $\mathcal{L}(X)$, defined by $\mathrm{E}(X_i) = X_i$ for $X_i \in X$ and $\mathrm{E}((b_1, b_2)) = [b_1, b_2]$ for $b_1, b_2 \in \mathrm{Br}(X)$.*

Definition 2.3 (Formal power series)**.** $\widehat{\mathcal{A}}(X)$ *denotes the (unital associative) algebra of* formal power series *generated by* $\mathcal{A}(X)$*. An element* $a \in \widehat{\mathcal{A}}(X)$ *is a sequence* $a = (a_n)_{n\in\mathbb{N}}$ *usually written* $a = \sum_{n\in\mathbb{N}} a_n$*, where* $a_n \in \mathcal{A}_n(X)$ *with, in particular,* $a_0 \in \mathbb{K}$ *being its constant term. We also define the Lie algebra of formal Lie series* $\widehat{\mathcal{L}}(X)$ *as the Lie algebra of formal power series* $a \in \widehat{\mathcal{A}}(X)$ *for which* $a_n \in \mathcal{L}(X)$ *for each* $n \in \mathbb{N}$*.*

For $S \in \mathcal{A}(X)$ with null constant term,

$$\exp(S) := \sum_{k=0}^{\infty} \frac{S^k}{k!} \quad \text{and} \quad \log(1+S) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} S^k \tag{2.3}$$

are well defined elements of $\widehat{\mathcal{A}}(X)$. Moreover, the following identities hold in $\widehat{\mathcal{A}}(X)$:

$$\exp(\log(1+S)) = 1+S \quad \text{and} \quad \log(\exp(S)) = S. \tag{2.4}$$

Definition 2.4 (Free nilpotent algebra)**.** *For* $N \in \mathbb{N}$*,*

$$\mathcal{A}^N(X) := \bigoplus_{n\in[\![0,N]\!]} \mathcal{A}_n(X), \tag{2.5}$$

is a linear subspace of $\mathcal{A}(X)$*, which is not a subalgebra of* $\mathcal{A}(X)$*. Let* $\pi_N : \widehat{\mathcal{A}}(X) \to \mathcal{A}^N(X)$ *be the canonical surjection (truncation map). The space* $\mathcal{A}^N(X)$ *can be given a structure of algebra by defining the multiplication of two elements* $a, b \in \mathcal{A}^N(X)$ *by* $\pi_N(ab)$ *i.e. the multiplication on* $\mathcal{A}^N(X)$ *is the same as on* $\mathcal{A}(X)$ *except that monomials of degree* $> N$ *are discarded. Then* π_N *is a morphism of algebras:*

$$\forall S, S' \in \widehat{\mathcal{A}}(X), \qquad \pi_N(SS') = \pi_N(S)\pi_N(S'). \tag{2.6}$$

Thus regarded, $\mathcal{A}^N(X)$ *is the free nilpotent associative algebra of order* $(N+1)$*, generated by* X *over the field* $\mathbb{K}$ *and the Lie subalgebra of* $\mathcal{A}^N(X)$ *spanned by* X *is*

$$\mathcal{L}^N(X) := \pi_N(\mathcal{L}(X)). \tag{2.7}$$

For $S \in \mathcal{A}^N(X)$ with null constant term,

$$\exp_N(S) := \pi_N(\exp(S)) \quad \text{and} \quad \log_N(1+S) := \pi_N(\log(1+S)) \tag{2.8}$$

are well defined elements of $\mathcal{A}^N(X)$. Moreover, the following identities hold in $\mathcal{A}^N(X)$:

$$\exp_N(\log_N(1+S)) = 1+S \quad \text{and} \quad \log_N(\exp_N(S)) = S \tag{2.9}$$

and for every $S \in \widehat{\mathcal{A}}(X)$,

$$\log_N(\pi_N(S)) = \pi_N(\log(S)). \tag{2.10}$$

2.2 A Lie group

We recall the celebrated Baker–Campbell–Hausdorff formula and its corollary on $\mathcal{A}^N(X)$, consequence of (2.10).

Proposition 2.5. *There exists a Lie series* $\mathrm{BCH}(A, B)$ *in two indeterminates* A, B *such that, for every* $P, Q \in \widehat{\mathcal{A}}(X)$ *with null constant terms, the following identity holds in* $\widehat{\mathcal{A}}(X)$*:*

$$\exp(P)\exp(Q) = \exp(\mathrm{BCH}(P, Q)). \tag{2.11}$$

For $N \in \mathbb{N}$ *and* $P, Q \in \mathcal{A}^N(X)$ *with null constant terms, the following identity holds in* $\mathcal{A}^N(X)$*:*

$$\exp_N(P)\exp_N(Q) = \exp_N(\mathrm{BCH}_N(P, Q)) \tag{2.12}$$

where $\mathrm{BCH}_N(P, Q) = \pi_N(\mathrm{BCH}(P, Q)) \in \mathcal{L}^N(X)$*.*

Example 2.6. *The formula up to order* 3 *is*

$$\mathrm{BCH}(A,B) = A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] + \frac{1}{12}[B,[B,A]] + \cdots \tag{2.13}$$

In particular, Proposition 2.5 implies that the sets

$$\widehat{G}(X) := \{\exp(Z); Z \in \widehat{\mathcal{L}}(X)\} \qquad \text{and} \qquad G^N(X) := \{\exp_N(Z); Z \in \mathcal{L}^N(X)\} \tag{2.14}$$

are respectively subgroups of $\widehat{\mathcal{A}}(X)$ and $\mathcal{A}^N(X)$. The elements of $\widehat{G}(X)$ are called exponential Lie series.

Proposition 2.7. *$G^N(X)$ is a Lie group, whose Lie algebra is $\mathcal{L}^N(X)$ i.e. $G^N(X)$ is a group and an analytic manifold, whose tangent space at* 1 *is $\mathcal{L}^N(X)$.*

2.3 A well-suited basis of the free Lie algebra

Designing a splitting method to compute the flow of $f_0 + f_1$ is, initially, a symmetric question with respect to f_0 and f_1. However, this symmetry breaks if one considers methods using only forward flows of f_0 (but both forward and backward flows of f_1). The resulting asymmetry is reflected in the formulation of the associated control-affine system (1.5). In order to identify the conditions on f_0 and f_1 under which high-order splitting methods exist, this asymmetry must be taken into account when choosing a basis of the free Lie algebra $\mathcal{L}(X)$.

The so-called (generalized) *Hall bases* (stemming from [32] and generalized in [56, 61]) constitute a wide family of bases of $\mathcal{L}(X)$, which includes many well-known bases of $\mathcal{L}(X)$ such as the historical length-compatible Hall bases of [32] or the Chen–Fox–Lyndon basis of [21]. We refer the interested reader to [4, Section 1.4] for a more gentle introduction and more thorough details.

Definition 2.8. *A* Hall set *over X is a subset $\mathcal{B}$ of* $\mathrm{Br}(X)$ *(see Definition 1.1), endowed with a total order $<$ such that:*

- *$X \subset \mathcal{B}$,*
- *for all $b_1, b_2 \in \mathrm{Br}(X)$, $(b_1, b_2) \in \mathcal{B}$ if and only if $b_1, b_2 \in \mathcal{B}$, $b_1 < b_2$ and, either $b_2 \in X$ or $b_2 = (b_3, b_4)$ with $b_3 \leq b_1$,*
- *for all $b_1, b_2 \in \mathcal{B}$ such that $(b_1, b_2) \in \mathcal{B}$ then $b_1 < (b_1, b_2)$.*

The importance of this definition is linked with the following result.

Theorem 2.9 (Viennot [61])**.** *Let $\mathcal{B} \subset \mathrm{Br}(X)$ be a Hall set. Then $\mathrm{E}(\mathcal{B})$ is a basis of $\mathcal{L}(X)$.*

In [7, Section 3], the first and third authors introduced a new Hall set $\mathcal{B}_\star$, specifically designed for applications to control theory, which correctly reflects the asymmetry between X_0 and X_1 corresponding to the control system (1.5). As we aim to illustrate in the following sections, this basis is also useful in the context of splitting methods. Since we will not need the full Hall set $\mathcal{B}_\star$ in the sequel, we omit the precise definition of the underlying order (for which we refer to [7, Section 3]), and we instead list below its elements of length at most 5, in the order under which they appear in $\mathcal{B}_\star$:

$$\begin{aligned} X_1 &< M_1 < M_2 < M_3 < M_4 \\ &< W_1 < (W_1, X_0) < ((W_1, X_0), X_0) < W_2 \\ &< (X_1, W_1) < ((X_1, W_1), X_0) < ((X_1, X_0), W_1) \\ &< Q_1 \\ &< X_0 \end{aligned} \tag{2.15}$$

where we use the notations of Definition 1.2 and

$$Q_1 := (X_1, (X_1, (X_1, (X_1, X_0)))) = \mathrm{ad}^4_{X_1}(X_0). \tag{2.16}$$

Although the 14 elements of (2.15) will be sufficient for our purpose here, the structure of the elements of $\mathcal{B}_\star$ is understood and easily computable for many more elements. In particular, it contains all the elements of the form M_ν for $\nu \in \mathbb{N}$ and W_j for $j \in \mathbb{N}^*$ defined in Definition 1.2.

3 The formal differential equation

3.1 The classical framework: integrable controls

3.1.1 An equation on $\widehat{\mathcal{A}}(X)$

Let $u = (u_0, \dots, u_m) \in L^1(\mathbb{R}^+, \mathbb{K}^{m+1})$. In this section, we consider the following formal differential equation set on $\widehat{\mathcal{A}}(X)$

$$\begin{cases} \dot{S}(t) = S(t) \sum_{j=0}^m u_j(t) X_j, \\ S(0) = 1, \end{cases} \tag{3.1}$$

whose solutions are defined in the following way.

Definition 3.1 (Solution to a formal differential equation)**.** *Let $u = (u_0, \dots, u_m) \in L^1(\mathbb{R}^+; \mathbb{K}^{m+1})$. The* solution to the formal differential equation (3.1) *is the formal-series valued function $S : \mathbb{R}^+ \to \widehat{\mathcal{A}}(X)$, whose homogeneous components $S_n : \mathbb{R}^+ \to \mathcal{A}_n(X)$ are the unique continuous functions that satisfy, for every $t \geq 0$, $S_0(t) = 1$ and, for every $n \in \mathbb{N}^*$,*

$$S_n(t) = \int_0^t S_{n-1}(\tau) \sum_{j=0}^m u_j(\tau) X_j \, \mathrm{d}\tau. \tag{3.2}$$

We denote by $\mathrm{Ser}(t, X, u)$ *this solution.*

Iterating this integral formula yields a power series expansion of $\mathrm{Ser}(t, X, u)$ in $\widehat{\mathcal{A}}(X)$ called the Chen series, [19, 20] and popularized in control theory by [28]. In this work, we will instead mostly work with the logarithm of the flow, using the Magnus formula.

For instance, if the u_j are piecewise constant functions with disjoint supports, then $\mathrm{Ser}(t, X, u)$ takes the following form, where $t_1, \dots, t_k \in \mathbb{R}$,

$$\exp(t_1 X_{j_1}) \exp(t_2 X_{j_2}) \dots \exp(t_k X_{j_k}).$$

The aim of the next proposition is to list the properties of the map $\mathrm{Ser}(\cdot, X, \cdot)$ that we will be using in this article. We will then need the following definitions.

Definition 3.2 (Concatenation of functions)**.** *For $u \in L^1((0, T), \mathbb{K})$ and $\widetilde{u} \in L^1((0, \widetilde{T}), \mathbb{K})$, we denote by $u \diamond \widetilde{u} : (0, T + \widetilde{T}) \to \mathbb{K}$ their concatenation*

$$(u \diamond \widetilde{u})(t) := \begin{cases} u(t) & \text{if } t \in (0, T), \\ \widetilde{u}(t - T) & \text{if } t \in (T, \widetilde{T}). \end{cases} \tag{3.3}$$

Definition 3.3 (Monomial basis of $\mathcal{L}(X)$)**.** *A monomial basis of $\mathcal{L}(X)$ is a basis of $\mathcal{L}(X)$ whose elements are iterated Lie brackets of elements of X (evaluations through* E *of elements of* $\mathrm{Br}(X)$*).*

Proposition 3.4. *The following hold:*

1. *For all* $u \in L^1((0,T),\mathbb{K}^{m+1})$ *and* $\widetilde{u} \in L^1((0,\widetilde{T}),\mathbb{K}^{m+1})$ *the following equality holds in* $\widehat{\mathcal{A}}(X)$
$$\mathrm{Ser}(T+\widetilde{T},X,u\diamond\widetilde{u}) = \mathrm{Ser}(T,X,u)\,\mathrm{Ser}(\widetilde{T},X,\widetilde{u}). \tag{3.4}$$

2. *For any* $u \in L^1(\mathbb{R}^+,\mathbb{K}^{m+1})$ *and* $t>0$, $\mathrm{Ser}(t,X,u)$ *is an exponential Lie series :* $\mathrm{Ser}(t,X,u) \in \widehat{G}(X)$ *i.e.* $\mathrm{Ser}(t,X,u) = e^{Z(t,X,u)}$ *where*
$$Z(t,X,u) := \log(\mathrm{Ser}(t,X,u)) \in \widehat{\mathcal{L}}(X). \tag{3.5}$$

3. *If* $\mathcal{B}$ *is a monomial basis of* $\mathcal{L}(X)$*, there exists a unique set of functionals* $(\zeta_b)_{b\in\mathcal{B}}$*, with* $\zeta_b : \mathbb{R}^+ \times L^1(\mathbb{R}^+,\mathbb{K}^{m+1}) \to \mathbb{K}$*, such that, for every* $u \in L^1(\mathbb{R}^+,\mathbb{K}^{m+1})$*,*
$$Z(t,X,u) = \sum_{b\in\mathcal{B}} \zeta_b(t,u) b. \tag{3.6}$$
These functionals are called coordinates of the first kind *associated to the basis* $\mathcal{B}$*.*

4. *The coordinates of the first kind enjoy the following homogeneity. Let* $\bar{u} : (0,T) \to \mathbb{K}^{m+1}$*. For* $\varepsilon > 0$*, let* $u^\varepsilon : t \in (0,\varepsilon T) \mapsto \bar{u}(t/\varepsilon)$*. Then, for every* $b \in \mathcal{B}$*,*
$$\zeta_b(\varepsilon T, u^\varepsilon) = \varepsilon^{|b|}\zeta_b(1,\bar{u}). \tag{3.7}$$
For $\lambda_0,\dots,\lambda_m \in \mathbb{K}$*, let* $u^\lambda := (\lambda_0\bar{u}_0,\dots,\lambda_m\bar{u}_m)$*. Then, for every* $b\in\mathcal{B}$*,*
$$\zeta_b(T,u) = \lambda_0^{n_0(b)}\cdots\lambda_m^{n_m(b)}\zeta_b(T,\bar{u}), \tag{3.8}$$
where the $n_j(b)$ *are defined in Definition 1.1.*

Proof of Proposition 3.4. The first statement is a consequence of the uniqueness, for any initial condition in $\widehat{\mathcal{A}}(X)$, of the solution of the formal differential equation (3.1). The second statement is a consequence of the Baker–Campbell–Hausdorff formula (see Proposition 2.5) when u is a piecewise constant function. In the general case $u \in L^1(\mathbb{R}^+,\mathbb{K}^{m+1})$, the Magnus formula gives the result, see for instance [5, Section 2.3]. The homogeneity of the coordinates of the first kind follows from (3.2). □

3.1.2 An equation on $\mathcal{A}^N(X)$

Equation (3.1) can also be considered as a differential equation on $\mathcal{A}^N(X)$. Then its solution is
$$\mathrm{Ser}_N(t,X,u) := \pi_N\left(\mathrm{Ser}(t,X,u)\right). \tag{3.9}$$
For instance, if the u_j are piecewise constant functions with disjoint supports, then $\mathrm{Ser}_N(t,X,u)$ takes the following form, where $t_1,\dots,t_k \in \mathbb{R}$,
$$\exp_N(t_1 X_{j_1})\dots\exp_N(t_k X_{j_k}).$$
The map $\mathrm{Ser}_N(\cdot,X,\cdot)$ has the following properties.

Proposition 3.5. *The following hold:*

1. *For all* $u \in L^1((0,T),\mathbb{K}^{m+1})$ *and* $\widetilde{u} \in L^1((0,\widetilde{T}),\mathbb{K}^{m+1})$ *the following equality holds in* $\mathcal{A}^N(X)$
$$\mathrm{Ser}_N(T+\widetilde{T},X,u\diamond\widetilde{u}) = \mathrm{Ser}_N(T,X,u)\,\mathrm{Ser}_N(\widetilde{T},X,\widetilde{u}). \tag{3.10}$$

2. *For every* $u \in L^1(\mathbb{R}^+,\mathbb{K})$ *and* $t>0$, $\mathrm{Ser}_N(t,X,u) \in G^N(X)$*. More precisely,* $\mathrm{Ser}_N(t,X,u) = \exp_N(Z_N(t,X,u))$ *where*
$$Z_N(t,X,u) := \pi_N Z(t,X,u) \in \mathcal{L}^N(X) \tag{3.11}$$

3. *If $\mathcal{B}$ is a monomial basis of $\mathcal{L}(X)$, then, for every $u \in L^1(\mathbb{R}^+, \mathbb{K}^{m+1})$,*

$$Z_N(t, X, u) = \sum_{b \in \mathcal{B}_N} \zeta_b(t, u) b \quad \text{where} \quad \mathcal{B}_N := \{b \in \mathcal{B}; |b| \leq N\} \tag{3.12}$$

Proof. The first statement is a consequence of the first statement of Proposition 3.4 and identity (2.6). The second statement is a consequence of the formula (2.10) applied to $S = \mathrm{Ser}(t, X, u)$. □

3.2 A new framework: Dirac controls

Let $X = \{X_0, X_1\}$, $m \in \mathbb{N}^*$, $c_1, \ldots, c_m \in \mathrm{Br}(X)$. We consider the formal differential equation

$$\begin{cases} \dot{S}(t) = S(t) \left(X_0 + \sum_{j=1}^m u_j(t) c_j \right), \\ S(0) = 1. \end{cases} \tag{3.13}$$

3.2.1 An equation on $\widehat{A}(X)$

As in the previous section, if $u := (u_1, \ldots, u_m) \in L^1(\mathbb{R}^+, \mathbb{K}^m)$, there exists a unique classical solution whose homogeneous components are continuous. We denote it by $\mathrm{Ser}(t, X, u)$. Our goal is to consider controls $u_1, \ldots, u_m$ that are finite sums of Dirac masses with (two by two) disjoint supports and real amplitudes. To simplify notations, we denote by $\mathcal{U}$ the set of such controls $u = (u_1, \ldots, u_m)$.

Definition 3.6 (Solution to (3.13))**.** *Let $u \in \mathcal{U}$ with $u_j = \sum_{k=1}^n a_k^j \delta_{\tau_k^j}$. For $\varepsilon > 0$, we define $u_\varepsilon := (\sum_{k=1}^n a_k^j 1_{[\tau_k^j, \tau_k^j + \varepsilon]})_{1 \leq j \leq m} \in L^1(\mathbb{R}^+, \mathbb{R}^m)$. The solution to the formal differential equation* (3.13) *is the formal series valued map $S : \mathbb{R}^+ \to \widehat{A}(X)$ whose homogeneous components $S_n : \mathbb{R}^+ \to \mathcal{A}_n(X)$ are the limit as $\varepsilon \to 0$ of the ones of* $\mathrm{Ser}(t, X, u_\varepsilon)$. *We denote it* $\mathrm{Ser}(t, X, u)$.

There is no ambiguity in this definition because the supports of the u_j are (two by two) disjoint. For instance, if we consider the formal differential equation

$$\begin{cases} \dot{S}(t) = S(t) \left(X_0 + u_1(t) X_1 + u_2(t) W_1 \right), \\ S(0) = 1, \end{cases} \tag{3.14}$$

and the controls $u_1 = 7\delta_3$ and $u_2 = 8\delta_5$ then the solution is

$$\mathrm{Ser}(t, X, u) = \begin{cases} \exp(tX_0) & \text{if } t \in (0, 3) \\ \exp((t-3)X_0) \exp(7X_1) \exp(3X_0) & \text{if } t \in (3, 5) \\ \exp((t-5)X_0) \exp(8W_1) \exp(2X_0) \exp(7X_1) \exp(3X_0) & \text{if } t \in (5, \infty). \end{cases}$$

Proposition 3.7. *The following hold:*

1. *If $u, \widetilde{u} \in \mathcal{U}$,* $\mathrm{supp}(u) \subset [0, T]$ *and* $\mathrm{supp}(\widetilde{u}) \subset [0, \widetilde{T}]$ *then*

$$\mathrm{Ser}(T + \widetilde{T}, X, u \diamond \widetilde{u}) = \mathrm{Ser}(T, X, u) \, \mathrm{Ser}(\widetilde{T}, X, \widetilde{u}). \tag{3.15}$$

2. *If $u \in \mathcal{U}$ and $t > 0$, $\mathrm{Ser}(t, X, u)$ is an exponential Lie series : $\mathrm{Ser}(t, X, u) \in \widehat{G}(X)$ i.e. $\mathrm{Ser}(t, X, u) = e^{Z(t, X, u)}$ where $Z(t, X, u) := \log(\mathrm{Ser}(t, X, u)) \in \widehat{\mathcal{L}}(X)$.*

3. *If $\mathcal{B}$ is a monomial basis of $\mathcal{L}(X)$, there exists a unique set of functionals $(\zeta_b)_{b \in \mathcal{B}}$, with $\zeta_b : \mathbb{R}^+ \times \mathcal{U} \to \mathbb{K}$, such that, for every $u \in \mathcal{U}$, $Z(t, X, u) = \sum_{b \in \mathcal{B}} \zeta_b(t, u) b$. These functionals are called* coordinates of the first kind *associated to the basis $\mathcal{B}$.*

4. *If $m = 1$ and $c_1 = X_1$, they have the same homogeneity properties as in Proposition 3.4.*

Proof. When $u, \widetilde{u} \in L^1(\mathbb{R}^+, \mathbb{K}^{m+1})$ then (3.15) results from the uniqueness of the solution of the Cauchy problem. When $u, \widetilde{u} \in \mathcal{U}$ then (3.15) is obtained by passing to the limit $\varepsilon \to 0$ (see (3.13)). The second statement is a consequence of the Baker–Campbell–Hausdorff formula (see Proposition 2.5). □

3.2.2 An equation on $\mathcal{A}^N(X)$

Equation (3.13) can also be considered as a differential equation on $\mathcal{A}^N(X)$. Then its solution is $\mathrm{Ser}_N(t,X,u) := \pi_N\left(\mathrm{Ser}(t,X,u)\right)$.

For instance, if we consider the formal differential equation (3.14) with the controls $u_1 = 7\delta_3$ and $u_2 = 8\delta_5$ then the solution is

$$\mathrm{Ser}_N(t,X,u) = \begin{cases} \exp_N(tX_0) & \text{if } t \in (0,3) \\ \exp_N((t-3)X_0)\exp_N(7X_1)\exp_N(3X_0) & \text{if } t \in (3,5) \\ \exp_N((t-5)X_0)\exp_N(8W_1)\exp_N(2X_0)\exp_N(7X_1)\exp_N(3X_0) & \text{if } t \in (5,\infty). \end{cases}$$

Proposition 3.8. *1. If $u, \widetilde{u} \in \mathcal{U}$, $\mathrm{supp}(u) \subset [0,T]$ and $\mathrm{supp}(\widetilde{u}) \subset [0,\widetilde{T}]$ then*

$$\mathrm{Ser}_N(T+\widetilde{T}, X, u \diamond \widetilde{u}) = \mathrm{Ser}_N(T,X,u)\,\mathrm{Ser}_N(\widetilde{T},X,\widetilde{u}). \tag{3.16}$$

2. If $u \in \mathcal{U}$ and $t > 0$, $\mathrm{Ser}_N(t,X,u) \in G^N(X)$. Moreover $\mathrm{Ser}_N(t,X,u) = \exp_N(Z_N(t,X,u))$ where $Z_N(t,X,u) = \pi_N Z(t,X,u) \in \mathcal{L}^N(X)$.

3. If $\mathcal{B}$ is a monomial basis of $\mathcal{L}(X)$, then, for every $u \in \mathcal{U}$, $Z_N(t,X,u) = \sum_{b \in \mathcal{B}_N} \zeta_b(t,u) b$ where $\mathcal{B}_N := \{b \in \mathcal{B}; |b| \leq N\}$.

The proof is similar to that of Proposition 3.5.

3.3 Lazard elimination and coordinates of the second kind

In this section, we define a generalization of the well-known Lazard elimination process and associated coordinates of the second kind, in the context of a multi-controlled system (3.17). Let $X = \{X_0, X_1\}$ and $\mathcal{B} \subset \mathrm{Br}(X)$ is a Hall set with X_0 maximal. Let $\mathcal{C}$ be a finite subset of $\mathcal{B} \setminus \{X_0\}$. We consider the formal differential equation

$$\begin{cases} \dot{S}(t) = S(t)\left(X_0 + \sum_{c \in \mathcal{C}} u_c(t) c\right) \\ S(0) = 1 \end{cases} \tag{3.17}$$

where the u_c are real-valued controls. One could also say that $X_0 \in \mathcal{C}$ with a fixed control $u_{X_0} \equiv 1$.

Given $b \in \mathcal{B}$, we denote in the following statements by $\mathbb{P}_{>b}$ the projection on $\mathrm{span}\{c \in \mathcal{B}; c > b\}$ parrallel to $\mathrm{span}\{c \in \mathcal{B}; c \leq b\}$ in $\mathcal{L}(X)$.

Definition 3.9 (Coordinates of the second kind)**.** *The coordinates of the second kind associated to the couple $(\mathcal{B}, \mathcal{C})$ is the unique family $(\xi_b)_{b \in \mathcal{B}}$ of functionals $\mathbb{R}^+ \times L^1 \to \mathbb{R}$ defined by induction in the following way: for any $u \in L^1$, $t \geq 0$ and $b \in \mathcal{B}$,*

$$\xi_b(t,u) = \sum \langle \mathrm{ad}_{b_r}^{m_r} \mathbb{P}_{>b_r} \cdots \mathrm{ad}_{b_1}^{m_1} \mathbb{P}_{>b_1}(c), b \rangle_{\mathcal{B}} \int_0^t \frac{\xi_{b_r}^{m_r}(s,u) \cdots \xi_{b_1}^{m_1}(s,u)}{m_r! \cdots m_1!} u_c(s)\,\mathrm{d}s, \tag{3.18}$$

where the sum ranges over $c \in \mathcal{C} \cup \{X_0\}$ (with the convention that $u_{X_0} \equiv 1$), $r \in \mathbb{N}$, $m_1, \dots, m_r \in \mathbb{N}^*$ and $b_1 < \cdots < b_r < b \in \mathcal{B}$.

Remark 3.10. *For any $b \in \mathcal{B}$, the sum (3.18) is finite since $\langle \mathrm{ad}_{b_r}^{m_r} \mathbb{P}_{>b_r} \cdots \mathrm{ad}_{b_1}^{m_1} \mathbb{P}_{>b_1}(c), b \rangle_{\mathcal{B}} \neq 0$ implies that $m_i |b_i| < |b|$.*

Proposition 3.11. *Let $N \in \mathbb{N}^*$. For any $u \in L^1$ and $t \geq 0$, the solution to (3.17) satisfies*

$$\pi_N(S(t)) = \overleftarrow{\prod_{b \in \mathcal{B}^N}} \exp_N(\xi_b(t,u) b) \tag{3.19}$$

where $\mathcal{B}^N := \{b \in \mathcal{B}; |b| \leq N\}$, and the terms in the product are ordered according to the order of the Hall set $\mathcal{B}$.

Proof. This statement is analogous to well-known results concerning the case of a single control u_{X_1} (or of multiple controls in front of multiple independent letters X_i). We explain what needs to be changed in the case of (3.17) when controls are placed in front of brackets of $\mathcal{B}$ (not only letters). Write $\mathcal{B}^N = \{b_1, \dots, b_R\}$ for some $R = |\mathcal{B}^N| \geq 2$ and $b_1 < \dots < b_R$. Our goal is to prove

$$\pi_N(S(t)) = \exp_N(\xi_{b_R}(t,u)b_R) \dots \exp_N(\xi_{b_1}(t,u)b_1).$$

We define $S_0 = S$ and, for $j \geq 1$

$$S_j(t) = S_{j-1}(t) \exp_N\left(\xi_{b_j}(t,u)b_j\right). \tag{3.20}$$

Let $\mathcal{C}_0 := \mathcal{C} \cup \{X_0\}$. We check by induction on $j \geq 0$ that, by (3.18),

$$\dot{S}_j(t) = S_j(t)\left(\sum_{m_1,\dots,m_j \in \mathbb{N}} \sum_{c \in \mathcal{C}_0} \frac{\xi_{b_j}^{m_j}(t,u)\cdots\xi_{b_1}^{m_1}(t,u)}{m_j!\cdots m_1!} u_c(t) \cdot \operatorname{ad}_{b_j}^{m_j} \mathbb{P}_{>b_j} \cdots \operatorname{ad}_{b_1}^{m_1} \mathbb{P}_{>b_1}(c)\right) \tag{3.21}$$

Thus, in particular, $\pi_N S_R(t) = 1$. Hence, $\pi_N(S(t))$ is given by the finite product of exponentials $\exp_N(\xi_{b_R}(t,u)b_R) \cdots \exp_N(\xi_{b_1}(t,u)b_1)$. □

Remark 3.12. *For L^1 controls, the integrands in* (3.18) *are well-defined and the ξ_b are continuous functions of time. For controls which are finite sums of Dirac masses with disjoint supports, it is straightforward to check that the ξ_b will be piecewise continuous. Unfortunately, this is not sufficient for the integrands in ξ_b to be well-defined, since the product $\xi_{b_r}^{m_r} \cdots \xi_{b_1}^{m_1}$ could be discontinuous on a Dirac mass of u_c. Hence, one should choose a regularization and use it to compute the coordinates. Their limit is unique because $S(t)$ is well-defined for such controls and obtained as a limit of regularizations, and by identification in* (3.19)*.*

Example 3.13. *In our basis $\mathcal{B}_\star$, for $k \geq 1$, one has*

$$\xi_{M_k}(t,u) = \int_0^t \left(\xi_{M_{k-1}}(s,u) + u_{M_k}(s)\right) \mathrm{d}s. \tag{3.22}$$

Moreover, if $\mathcal{C}$ does not involve $\{M_1, \dots, M_{2k-1}\}$,

$$\xi_{W_k}(t,u) = \int_0^t \left(\frac{1}{2}\xi_{M_{k-1}}^2(s,u) + u_{W_k}(s)\right) \mathrm{d}s. \tag{3.23}$$

3.4 Coordinates of the first and second kind are diffeomorphic

Proposition 3.14. *Let $N \in \mathbb{N}^*$ and $d := \dim(\mathcal{L}^N(X))$. There exists a global diffeomorphism Φ of $\mathbb{R}^d$ such that, for every $t > 0$ and $u \in \mathcal{U}$, $(\zeta_b(t,u))_{b \in \mathcal{B}^N} = \Phi\left((\xi_b(t,u))_{b \in \mathcal{B}^N}\right)$. More precisely, for every $b \in \mathcal{B}$, there exists a polynomial P_b such that*

$$\zeta_b(t,u) = \xi_b(t,u) + P_b((\xi_{b'}(t,u))_{|b'|<|b|}), \tag{3.24}$$

where P_b only involves elements $b' \in \operatorname{Br}(X)$ such that $n_i(b') \leq n_i(b)$ with $i \in \{0, 1, \dots, m\}$ (with at least one strict inequality).

Proof. We apply the Baker–Campbell–Hausdorff formula (see Proposition 2.5) to the product

$$\overleftarrow{\prod_{b \in \mathcal{B}^N}} \exp_N(\xi_b(t,u)b) = \operatorname{Ser}_N(t,X,u) = \exp_N\left(\sum_{b \in \mathcal{B}^N} \zeta_b(t,u)b\right).$$

□

For example, the first terms of the BCH formula (2.13) prove the following explicit expressions, that will be used in Section 7.1: $\zeta_{X_0}(t,u)=\xi_{X_0}(t,u)$, $\zeta_{X_1}(t,u)=\xi_{X_1}(t,u)$ and

$$\begin{aligned}
\zeta_{M_1}(t,u) &= \xi_{M_1}(t,u)-\frac{1}{2}\xi_{X_0}(t,u)\xi_{X_1}(t,u),\\
\zeta_{M_2}(t,u) &= \xi_{M_2}(t,u)-\frac{1}{2}\xi_{X_0}(t,u)\xi_{M_1}(t,u)+\frac{1}{12}\xi_{X_0}(t,u)^2\xi_{X_1}(t,u),\\
\zeta_{W_1}(t,u) &= \xi_{W_1}(t,u)-\frac{1}{2}\xi_{M_1}(t,u)\xi_{X_1}(t,u)+\frac{1}{12}\xi_{X_1}(t,u)^2\xi_{X_0}(t,u).
\end{aligned} \tag{3.25}$$

More generally, for every $n\in\mathbb{N}^*$, there exists real numbers $\gamma_{j,k}\in\mathbb{R}$ such that

$$\zeta_{W_n}(t,u)=\xi_{W_n}(t,u)+\sum_{\substack{j,k\in\mathbb{N}\\ j+k\leq 2n-1}}\gamma_{j,k}\xi_{M_j}(t,u)\xi_{M_k}(t,u)\xi_{X_0}(t,u)^{2n-1-j-k}.$$

4 Error estimates for the Magnus formula

4.1 Affine systems with integrable controls

In this section, we consider the control system

$$\dot{x}(t)=\sum_{j=0}^{m}u_j(t)f_j(x(t)) \tag{4.1}$$

where $m\in\mathbb{N}^*$, $u=(u_0,\dots,u_m)\in L^1(\mathbb{R}^+,\mathbb{K}^{m+1})$, $f_0,\dots,f_m$ are vector fields on $\mathbb{K}^d$. When well defined, the solution associated to the initial condition $x(0)=p$ is denoted $x(t;f,u,p)$.

Definition 4.1 ($Z_N(t,f,u)$ for $u\in L^1$)**.** *Let $t>0$ and $u=(u_0,\dots,u_m)\in L^1(\mathbb{R}^+,\mathbb{K}^{m+1})$. For $f_0,\dots,f_m\in\mathcal{C}^\infty(\mathbb{K}^d,\mathbb{K}^d)$, we define*

$$Z_N(t,f,u):=Ev_f(Z_N(t,X,u))$$

where $Ev_f:\mathcal{L}(X)\to\mathcal{C}^\infty(\mathbb{K}^d,\mathbb{K}^d)$ is the unique morphism of Lie algebras such that $Ev_f(X_j)=f_j$ for $j=0,\dots,m$ and $Z_N(t,X,u)$ is defined in Proposition 3.5. If $\mathcal{B}$ is a monomial basis of $\mathcal{L}(X)$, then

$$Z_N(t,f,u)=\sum_{b\in\mathcal{B}_N}\zeta_b(t,u)f_b$$

where the ζ_b are the coordinates of the first kind defined in Proposition 3.4. This definition can be extended to $\mathcal{C}^N$ vector fields on $\mathbb{K}^d$ defined locally and then $Z_N(t,f,u)$ is a $\mathcal{C}^1$ vector field defined locally.

The following estimate is proved in [5, Proposition 93].

Proposition 4.2. *Let $p\in\mathbb{K}^d$. For any $r>0$, B_r denotes the open ball $B_{\mathbb{K}^d}(p,r)$. For every $N\in\mathbb{N}$, there exists $\delta_N,C_N>0$ such that, for every $\delta>0$, $T>0$, $u\in L^1((0,T),\mathbb{K}^{m+1})$, $f_0,\dots,f_m\in\mathcal{C}^{N^2}(B_{2\delta},\mathbb{K}^d)$ with $\rho:=\sum_{j=0}^m\|u_j\|_{L^1}\|f_j\|_{\mathcal{C}^{N^2}}\leq\delta_N\min\{1;\delta\}$, and $t\in[0,T]$,*

$$\left|x(t;f,u,p)-e^{Z_N(t,f,u)}p\right|\leq C_N\rho^{N+1}. \tag{4.2}$$

The following estimate is an immediate consequence of Proposition 4.2 (see [5, Proposition 161] for a proof). Here, B_r denotes a ball centered at 0.

Proposition 4.3. *Let $N \in \mathbb{N}^*$ and $f_0 \dots, f_m$ be $\mathcal{C}^{M^2}$ vector fields on $\mathbb{K}^d$ defined on an open neighborhood of* 0*. The following estimate holds, as $T \to 0$,*

$$x(T; f, u, 0) = Z_N(T, f, u)(0) + O\left(T^{N+1} + T|x(T; f, u, 0)|\right). \tag{4.3}$$

in the following sense: there exist $C, \eta > 0$ such that, for every $T \in (0, \eta]$ and $u \in L^\infty((0,T); \mathbb{K}^{m+1})$ with $\|u\|_{L^\infty} \leq 1$,

$$|x(T; f, u, 0) - Z_N(T, f, u)(0)| \leq C\left(T^{N+1} + T|x(T; f, u, 0)|\right). \tag{4.4}$$

4.2 Affine systems with Dirac controls

Let $X = \{X_0, X_1\}$, $m, d \in \mathbb{N}^*$, $c_1, \dots, c_m \in \mathrm{Br}(X)$ and f_0, f_1 be smooth vector fields on $\mathbb{K}^d$. In this section, we consider systems of the form

$$\dot{x}(t) = f_0(x(t)) + \sum_{j=1}^{m} u_j(t) f_{c_j}(x(t)) \tag{4.5}$$

with state $x(t) \in \mathbb{K}^d$ and control $u = (u_1, \dots, u_m)$.

For a given $p \in \mathbb{R}^d$, if $T > 0$ and $u \in L^1((0,T), \mathbb{K}^m)$ are small enough, there exists a unique classical solution $x \in C^0((0,T), \mathbb{K}^d)$, associated with the initial condition $x(0) = p$. We denote it by $x(t; f, u, p)$.

Our goal is to consider $u_1, \dots, u_m$ that are finite sums of Dirac masses with (two by two) disjoint supports and amplitudes in $\mathbb{K}$. To simplify notations, we denote by $\mathcal{U}$ the set of such controls $u = (u_1, \dots, u_m)$ and $\|u\|_{\mathcal{U}} := \sum_{j=1}^k |a_j|$ when $u = \sum_{j=1}^k a_j \delta_{\tau_j}$ with $a_j \in \mathbb{K}^m$.

Definition 4.4. *Let $u \in \mathcal{U}$ with $u_j = \sum_{k=1}^n a_k^j \delta_{\tau_k^j}$. For $\varepsilon > 0$, we define the regularization $u_\varepsilon := (\sum_{k=1}^n a_k^j 1_{[\tau_k^j, \tau_k^j + \varepsilon]})_{1 \leq j \leq m} \in L^1(\mathbb{R}^+, \mathbb{R})$. Let $p \in \mathbb{R}^d$. The solution to* (4.5) *associated with the initial condition $x(0) = p$ is the limit as $\varepsilon \to 0$ of $x(t; f, u_\varepsilon, p)$.*

For instance, if we consider the formal differential equation

$$\dot{x} = f_0(x) + u_1 f_1(x) + u_2 f_{W_1}(x),$$

and the controls $u_1 = 7\delta_3$ and $u_2 = 8\delta_5$ then the solution is

$$x(t; f, u, p) = \begin{cases} e^{tf_0} p & \text{if } t \in (0,3) \\ e^{(t-3)f_0} e^{7f_1} e^{3f_0} p & \text{if } t \in (3,5) \\ e^{(t-5)f_0} e^{8f_{W_1}} e^{2f_0} e^{7f_1} e^{3f_0} p & \text{if } t \in (5, \infty) \end{cases}$$

provided each flow is well defined.

Definition 4.5 ($Z_N(t, f, u)$ for $u \in \mathcal{U}$)**.** *Let $t > 0$ and $u \in \mathcal{U}$. For $f_0, f_1 \in \mathcal{C}^\infty(\mathbb{K}^d, \mathbb{K}^d)$, we define*

$$Z_N(t, f, u) := Ev_f(Z_N(t, X, u))$$

where $Ev_f : \mathcal{L}(X) \to \mathcal{C}^\infty(\mathbb{K}^d, \mathbb{K}^d)$ is the unique morphism of Lie algebras such that $Ev_f(X_j) = f_j$ for $j = 0, 1$ and $Z_N(t, X, u)$ is defined in Proposition 3.8. If $\mathcal{B}$ is a monomial basis of $\mathcal{L}(X)$, then

$$Z_N(t, f, u) = \sum_{b \in \mathcal{B}_N} \zeta_b(t, u) f_b$$

where the ζ_b are the coordinates of the first kind defined in Proposition 3.8. This definition can be extended to $\mathcal{C}^N$ vector fields on $\mathbb{K}^d$ defined locally and then $Z_N(t, f, u)$ is a $\mathcal{C}^1$ vector field defined locally.

Proposition 4.6. *Let $p \in \mathbb{R}^d$. For any $r > 0$, B_r denotes the open ball $B_{\mathbb{K}^d}(p, r)$. For every $N \in \mathbb{N}$, there exists $\delta_N, C_N > 0$ such that, for every $\delta > 0$, $T > 0$, $u \in \mathcal{U}$, $f_0, f_1 \in \mathcal{C}^{N^2}(B_{2\delta}, \mathbb{K}^d)$ with $\rho := (T + \|u\|_{\mathcal{U}})\|f\|_{\mathcal{C}^{N^2}} \leq \delta_N \min\{1; \delta\}$, and $t \in [0, T]$,*

$$\left| x(t; f, u, p) - e^{Z_N(t,f,u)} p \right| \leq C_N \rho^{N+1}. \tag{4.6}$$

Proof. We consider the sequence $u^\varepsilon \in L^1((0, T), \mathbb{R}^m)$ as in Definition 4.4. They satisfy $\|u^\varepsilon\|_{L^1} = \|u\|_{\mathcal{U}}$. By the previous section, the estimate holds for u_ε. We pass to the limit $\varepsilon \to 0$ to conclude. □

The following estimate is an immediate consequence of Proposition 4.6. Here, B_r denotes a ball centered at 0.

Proposition 4.7. *Let $N \in \mathbb{N}^*$ and f_0, f_1 be $\mathcal{C}^{N^2}$ vector fields on $\mathbb{K}^d$ defined on an open neighborhood of 0. The following estimate holds, as $T \to 0$,*

$$x(T; f, u, 0) = Z_N(T, f, u)(0) + O\left((T + \|u\|_{\mathcal{U}})^{N+1} + (T + \|u\|_{\mathcal{U}})|x(T; f, u, 0)| \right) \tag{4.7}$$

in the following sense: there exist $C, \eta > 0$ such that, for every $T \in (0, \eta]$ and $u \in \mathcal{U}$ with $\|u\|_{\mathcal{U}} \leq 1$,

$$|x(T; f, u, 0) - Z_N(T, f, u)(0)| \leq C\left((T + \|u\|_{\mathcal{U}})^{N+1} + (T + \|u\|_{\mathcal{U}})|x(T; f, u, 0)| \right). \tag{4.8}$$

5 High-order theory with unsigned coefficients

5.1 Prerequisite: accessibility

Theorem 5.1. *Let $d \in \mathbb{N}^*$, $\mathcal{M}$ be an analytic submanifold of dimension d and $x_0 \in \mathcal{M}$. Let $m \in \mathbb{N}^*$ and $f_0, \dots, f_m$ be analytic vector fields on $\mathcal{M}$ such that $\mathcal{L}(f_0, \dots, f_m)(x_0) = T_{x_0}\mathcal{M}$. There exists $j_1, \dots, j_d \in \{0, \dots, m\}$ and $\mathfrak{t}^0 \in (0, \infty)^d$ such that the differential at $\mathfrak{t}^0$ of the map*

$$F : \begin{cases} \mathbb{R}^d & \to \mathcal{M}, \\ \mathfrak{t} = (t_1, \dots, t_d) & \mapsto e^{t_d f_{j_d}} \cdots e^{t_1 f_{j_1}}(x_0). \end{cases} \tag{5.1}$$

has rank d. Moreover, j_1 can be any element of $\{0, \dots, m\}$ such that $f_{j_1}(x_0) \neq 0$, and $\mathfrak{t}^0$ can be arbitrary small.

Proof. If $\mathcal{M} = \mathbb{R}^d$, the first statement is proved in [24, Theorem 3.19] for $\mathcal{C}^\infty$ vector fields. Then one obtains $\mathfrak{t}^* \in \mathbb{R}^d$ such that $\det(DF(\mathfrak{t}^*)) \neq 0$. If the vector fields are analytic, then the map $\mathfrak{t} \mapsto \det(DF(\mathfrak{t}))$ is analytic not identically zero. Thus, for every $\delta > 0$, there exists $\mathfrak{t}^0 \in (0, \delta)^k$ such that $\det(DF(\mathfrak{t}^0)) \neq 0$. Finally, if $\mathcal{M}$ is an analytic manifold, we conclude with a local map. □

This theorem applies in particular to the formal differential equation on $\mathcal{A}^N(X)$ with $\mathcal{M} = G^N(X)$ and $f_j(S) = SX_j$ for $j = 0, \dots, m$, $x_0 = 1$ which gives the following statement.

Proposition 5.2. *Let $N, m \in \mathbb{N}^*$, $X = \{X_0, \dots, X_m\}$ and $k := \dim(\mathcal{L}^N(X))$. There exist $j_1, \dots, j_k \in \{0, \dots, m\}$ and $\mathfrak{t}^0 \in (0, \infty)^k$ such that the differential at $\mathfrak{t}^0$ of the map*

$$\begin{array}{rcl} \mathbb{R}^k & \to & G^N(X) \\ \mathfrak{t} = (t_1, \dots, t_k) & \mapsto & \exp_N(t_1 X_{j_1}) \dots \exp_N(t_k X_{j_k}). \end{array}$$

has rank equal to the dimension of $G^N(X)$. Moreover, j_1 can be any element of $\{0, \dots, m\}$ and $\mathfrak{t}^0$ can be arbitrary small.

5.2 An improved local inversion on $G^N(X)$

In this section, $N \in \mathbb{N}^*$ is fixed. For $k \in \mathbb{N}^*$, we define the map

$$\left| \begin{array}{llll} \nu_k^N : & \mathbb{R}^k & \mapsto & G^N(X) \\ & \mathfrak{t} = (t_1, \dots, t_k) & \mapsto & \exp_N(t_1 X_0) \exp_N(t_2 X_1) \exp_N(t_3 X_0) \dots \exp_N(t_k X_{\epsilon_k}) \end{array} \right.$$

where $\epsilon_k := 0$ if k is odd, 1 if k is even.

Definition 5.3 ($u^{\mathfrak{t}}$)**.** *For every $\mathfrak{t} \in \mathbb{R}^k$, there exist unique piecewise constant controls with disjoint supports $u_0^{\mathfrak{t}}, u_1^{\mathfrak{t}} : (0, |\mathfrak{t}|) \to \{-1, 0, 1\}$ such that $\nu_k^N(\mathfrak{t}) = \mathrm{Ser}_N(|\mathfrak{t}|, X, u^{\mathfrak{t}})$ where $u^{\mathfrak{t}} := (u_0^{\mathfrak{t}}, u_1^{\mathfrak{t}})$, and thus $\log_N(\nu_k^N(\mathfrak{t})) = Z_N(|\mathfrak{t}|, X, u^{\mathfrak{t}})$ (see Section 3.1).*

Definition 5.4 (Good element of $G^N(X)$)**.** *A element $S \in G^N(X)$ is good if there exists $k \in \mathbb{N}^*$ and $\mathfrak{t}^0 = (t_1^0, \dots, t_k^0) \in \mathbb{R}^k$, such that, $\nu_k^N(\mathfrak{t}^0) = S$ and $d\nu_k^N(\mathfrak{t}^0)$ has rank equal to the dimension of $G^N(X)$.*

Then by the inverse mapping theorem, there exists an open neighborhood of S in $G^N(X)$ on which ν_k^N has a $\mathcal{C}^1$ right inverse. In other words, there exists a neighborhood Ω of $Z := \log_N(S)$ in $\mathcal{L}^N(X)$ and a $\mathcal{C}^1$ map $\psi : \Omega \to \mathbb{R}^k$ such that, for every $Z' \in \Omega$, $Z' = Z_N(|\mathfrak{t}|, X, u^{\mathfrak{t}})$ for $\mathfrak{t} = \psi(Z')$.

By Definition 5.4 and Proposition 3.5, if S is a good element of $G^N(X)$ then any element of $G^N(X)$ obtained from S by right-multiplication with an element of the form $\nu_{k',v'}^N(\mathfrak{t}')$ is also a good element of $G^N(X)$.

Proposition 5.5. *1 is a good element of $G^N(X)$. Moreover, Definition 5.4 holds with $k = 2\dim(\mathcal{L}^N(X)) - 1$.*

Proof. Proposition 5.2 with $\mathbb{K} = \mathbb{R}$ provides $\mathfrak{t}^0 = (t_1^0, \dots, t_k^0) \in (0, \infty)^k$ with $k := \dim(\mathcal{L}^N(X))$ such that $\nu_k^N(\mathfrak{t}^0)$ be a good element of $G^N(X)$. Let $\mathfrak{t} \in \mathbb{R}^{2k-1}$ be defined by

$$\mathfrak{t} = (t_1^0, t_2^0, \dots, t_k^0 + (-1)^{k+1} t_k^0, \dots, -t_2^0, t_1^0),$$

i.e. $u^{\mathfrak{t}} : (0, 2T) \to \mathbb{R}$ is the concatenation of $u^{\mathfrak{t}^0}$ and $-\check{u}^{\mathfrak{t}^0}$, with $T := |\mathfrak{t}^0|$. By uniqueness of the solution of the formal differential equation on $\mathcal{A}^N(X)$, it satisfies $S(T+t) = S(T-t)$ for every $t \in (0, T)$, in particular $S(2T) = S(0) = 1$ i.e. $\nu_{2k-1}^N(\mathfrak{t}) = 1$. Moreover $\nu_{2k-1}^N(\mathfrak{t}) = \nu_k^N(\mathfrak{t}^0)\,\mathrm{Ser}_N(T, -\check{u}^{\mathfrak{t}^0})$ thus 1 is a good element of $G^N(X)$. ☐

5.3 Arbitrary order splitting methods with real-valued coefficients

The goal of this section is to prove Theorem 1.9. By Proposition 4.2, it suffices to prove the following statement.

Theorem 5.6. *Let $X = \{X_0, X_1\}$, $\mathbb{K} = \mathbb{R}$, $N \in \mathbb{N}^*$ and $k := \dim(\mathcal{L}^N(X))$. There exist $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_{k-1} \in \mathbb{R}$ such that the following equality holds in $G^N(X)$*

$$\exp_N(X_0 + X_1) = \exp_N(\alpha_1 X_0) \exp_N(\beta_1 X_1) \dots \exp_N(\beta_{k-1} X_1) \exp_N(\alpha_k X_0). \tag{5.2}$$

In other words, there exist piecewise constant controls $u_0, u_1 : (0,1) \to \mathbb{R}$ with disjoint supports such that $Z_N(1, X, u) = X_0 + X_1$, where Z_N is defined in Proposition 3.5 and $u = (u_0, u_1)$.

Proof. By Proposition 5.5, there exist a neighborhood Ω of 0 in $\mathcal{L}^N(X)$ and a $\mathcal{C}^1$ map $\psi : \Omega \to \mathbb{R}^{2k-1}$ such that, for every $Z \in \Omega$, $Z_N(|\psi(Z)|, X, u^{\psi(Z)}) = Z$. For $\eta > 0$ small enough, $\eta(X_0 + X_1) \in \Omega$. Let $\mathfrak{t} := \psi(\eta(X_0 + X_1)) = (t_1, \dots, t_{2k-1}) \in \mathbb{R}^{2k-1}$. Then the following equality holds in $\mathcal{A}^N(X)$

$$\exp_N(\eta(X_0 + X_1)) = \exp_N(t_1 X_0) \exp_N(t_2 X_1) \dots \exp_N(t_{2k-2} X_1) \exp_N(t_{2k-1} X_0).$$

By homogeneity, this gives the conclusion with $\alpha_j := t_{2j-1}/\eta$ and $\beta_j := t_{2j}/\eta$. ☐

5.4 Chow's theorem for control

We prove the implication of Theorem 1.10 stating that the Lie algebra rank condition implies controllability (precised in Proposition 5.7), to emphasize that it relies on the same argument (Proposition 5.5) as Theorem 1.9. For the other implication (that controllability implies the Lie algebra rank condition), see e.g. [24, Theorem 3.17] for a modern reference. In particular, analyticity is not required for Proposition 5.7 (but it is required for the other implication of Theorem 1.10).

Proposition 5.7. *Let f_0, f_1 be smooth vector fields on a neighborhood of 0 in $\mathbb{R}^d$ such that $\mathrm{Lie}_{\mathbb{R}}(f_0, f_1)(0) = \mathbb{R}^d$. Then, for every $\delta > 0$, there exists $r > 0$ such that, for every $x^* \in B_{\mathbb{R}^d}(0, r)$, there exists $T \in (0, \delta)$ and $u_0, u_1 : (0, T) \to \{-1, 0, 1\}$ piecewise constant with disjoint support such that $x(T; f, u, 0) = x^*$ for $u = (u_0, u_1)$.*

Proof. Let $\delta > 0$. By the Lie algebra rank condition, there exist $b_1 \dots, b_d \in \mathrm{Br}(X)$ such that $(f_{b_1}(0), \dots, f_{b_d}(0))$ be a basis of $\mathbb{R}^d$. Let $N := \max\{|b_j|; j \in [\![1, d]\!]\}$ and $k := 2\dim(\mathcal{L}^N(X)) - 1$. Let C, η be given by Proposition 4.3. One may assume $\delta < \eta$.

Let $\mathfrak{t}^0 \in \mathbb{R}^k$ be given by Proposition 5.5. By the inverse mapping theorem, there exist an open neighborhood Ω of 0 in $\mathcal{L}^N(X)$, an open neighborhood $\mathfrak{T}$ of $\mathfrak{t}^0$ in $(0, \infty)^k$ and a $\mathcal{C}^1$-map $\psi : \Omega \to \mathfrak{T}$ such that, for every $Z \in \Omega$, $Z_N(|\psi(Z)|, X, u^{\psi(Z)}) = Z$. One may assume Ω and $\mathfrak{T}$ small enough so that $|\psi(Z)| \leq 2|\mathfrak{t}^0|$ for every $Z \in \Omega$.

There exists $\rho^* > 0$ such that, for every $\rho = (\rho_1, \dots, \rho_d) \in \overline{B}_{\mathbb{R}^d}(0, \rho^*)$, we have $\sum_{j=1}^d \rho_j b_j \in \Omega$ and then $\mathfrak{t}_\rho := \psi(\sum_{j=1}^d \rho_j b_j)$ satisfies $|\mathfrak{t}_\rho| \leq 2|\mathfrak{t}^0|$ and $Z_N(|\mathfrak{t}_\rho|, X, u^{\mathfrak{t}_\rho}) = \sum_{j=1}^d \rho_j b_j$.

For $\rho \in B_{\mathbb{R}^d}(0, \rho^*)$ and $\varepsilon > 0$, we define

$$\begin{array}{rccl} u_\varepsilon^{\mathfrak{t}_\rho} : & (0, \varepsilon|\mathfrak{t}_\rho|) & \to & \{-1, 0, 1\}^2 \\ & t & \mapsto & u^{\mathfrak{t}_\rho}\left(\frac{t}{\varepsilon}\right). \end{array}$$

Then, using the homogeneity of the coordinates of the first kind (see Item 4 of Proposition 3.4),

$$Z_N(\varepsilon|\mathfrak{t}_\rho|, X, u_\varepsilon^{\mathfrak{t}_\rho}) = \sum_{j=1}^d \rho_j \varepsilon^{|b_j|} b_j. \tag{5.3}$$

We introduce

$$\alpha \in \left(\frac{1}{N+1}, \frac{1}{N}\right), \quad C' := \max\left\{\sum_{j=1}^d |\omega_j||f_{b_j}(0)|; \sum_{j=1}^d \omega_j f_{b_j}(0) \in \mathbb{S}^{d-1}\right\} \tag{5.4}$$

$$r := \min\left\{1; \left(\frac{\rho^*}{C'}\right)^{\frac{1}{1-\alpha N}}; \left(\frac{\delta}{2|\mathfrak{t}^0|}\right)^{\frac{1}{\alpha}}\right\}. \tag{5.5}$$

For $z = |z| \sum_{j=1}^d \omega_j f_{b_j}(0) \in B_{\mathbb{R}^d}(0, r)$, we define $\varepsilon(z) := |z|^\alpha$ and $\rho(z) := \sum_{j=1}^d |z|^{1-\alpha|b_j|} \omega_j f_{b_j}(0)$ (for $z = 0$, we take $\rho(z) = 0$ so that ρ be continuous). Then, by (5.4) and (5.5),

$$|\rho(z)| \leq |z|^{1-\alpha N} \sum_{j=1}^d |\omega_j||f_{b_j}(0)| \leq r^{1-\alpha N} C' \leq \rho^* \qquad \text{and} \qquad \varepsilon(z)|\mathfrak{t}_\rho| \leq 2r^\alpha |\mathfrak{t}^0| \leq \delta.$$

Thus the control $u_{\varepsilon(z)}^{\mathfrak{t}_{\rho(z)}}$ is well defined and its interval of definition is $\subset (0, \delta)$. To simplify the notation we write T_z instead of $\varepsilon(z)|\mathfrak{t}_{\rho(z)}|$ and u_z instead of $u_{\varepsilon(z)}^{\mathfrak{t}_{\rho(z)}}$. Then by (5.3)

$$Z_N(T_z, f, u_z)(0) = z.$$

Thus, by Proposition 4.3,

$$\begin{aligned}|x(T_z; f, u_z, 0) - z| &\leq C\left(T_z^{N+1} + T_z|x(T_z, f, u_z, 0)|\right) \\ &\leq C\left((2|z|^\alpha|\mathsf{t}^0|)^{N+1} + 2|\mathsf{t}^0||z|^\alpha|x(T_z, f, u_z, 0)|\right).\end{aligned}$$

This proves there exists $C'' > 0$ and $\beta > 1$ such that, for every $z \in \mathcal{B}_{\mathbb{R}^d}(0, \rho^*)$,

$$|x(T_z; f, u_z, 0) - z| \leq C'|z|^\beta.$$

Let $r' \in (0, r]$ be such that $C''(r')^\beta < r'/2$ and $x^* \in B_{\mathbb{R}^d}(0, r'/2)$. Then, the continuous map

$$\begin{array}{rccl} F: & \overline{B}_{\mathbb{R}^d}(0, r') & \to & \mathbb{R}^d \\ & z & \mapsto & z - x(T_z; f, u_z, 0) + x^* \end{array}$$

takes values in $\overline{B}_{\mathbb{R}^d}(0, r')$. By the Brouwer fixed point theorem, there exists $z \in \overline{B}_{\mathbb{R}^d}(0, r')$ such that $F(z) = z$, i.e. $x(T_z; f, u_z, 0) = x^*$. □

6 Complex controls and complex splitting methods

We prove Theorems 1.11 and 1.12 thanks to the same key ingredient of Proposition 6.4.

6.1 An improved local inversion on $\mathcal{G}^N(X)$

In this section, $X = \{X_0, X_1\}$, $\mathbb{K} = \mathbb{C}$. $\mathcal{L}(X)$ is also a vector space over the field $\mathbb{R}$. Let $\mathrm{Lie}_{\mathbb{R}}(X_0, X_1, iX_1)$ be the Lie subalgebra of $\mathcal{L}(X)$ generated by $\{X_0, X_1, iX_1\}$ over the field $\mathbb{R}$; it is also an hyperplane of $\mathcal{L}(X)$:

$$\mathcal{L}(X) = i\mathbb{R}X_0 \oplus \mathrm{Lie}_{\mathbb{R}}(X_0, X_1, iX_1).$$

In this section, $N \in \mathbb{N}^*$ is fixed. $\mathrm{Lie}^N_{\mathbb{R}}(X_0, X_1, iX_1) := \pi_N \mathrm{Lie}_{\mathbb{R}}(X_0, X_1, iX_1)$ is the Lie subalgebra of $\mathcal{L}^N(X)$ generated by $\{X_0, X_1, iX_1\}$ over $\mathbb{R}$. It is also an hyperplane of $\mathcal{L}^N(X)$:

$$\mathcal{L}^N(X) = i\mathbb{R}X_0 \oplus \mathrm{Lie}^N_{\mathbb{R}}(X_0, X_1, iX_1).$$

The BCH formula proves that the set

$$\mathcal{G}^N(X) := \{\exp_N(Z); Z \in \mathrm{Lie}^N_{\mathbb{R}}(X_0, X_1, iX_1)\}$$

is a subgroup of $\mathcal{A}^N(X)$.

Proposition 6.1. *$\mathcal{G}^N(X)$ is a Lie group, i.e. a group and an analytic manifold, whose tangent space at* 1 *is* $\mathrm{Lie}^N_{\mathbb{R}}(X_0, X_1, iX_1)$.

For $k \in \mathbb{N}^*$ and $v = (v_2, v_4, \dots, v_{2\lfloor \frac{k}{2} \rfloor}) \in \mathbb{U}^{\lfloor \frac{k}{2} \rfloor}$, we define the map $\nu^N_{k,v} : \mathbb{R}^k \to \mathcal{G}^N(X)$ by: for every $\mathsf{t} = (t_1, \dots, t_k) \in \mathbb{R}^k$,

$$\nu^N_{k,v}(\mathsf{t}) = \begin{cases} \exp_N(t_1 X_0) \exp_N(t_2 v_2 X_1) \dots \exp_N(t_k X_0) & \text{if } k \text{ is odd,} \\ \exp_N(t_1 X_0) \exp_N(t_2 v_2 X_1) \dots \exp_N(t_k v_k X_0) & \text{if } k \text{ is even.} \end{cases}$$

Definition 6.2 ($u^{\mathsf{t},v}$)**.** *For every* $\mathsf{t} \in (0, \infty)^k$ *and* $v \in \mathbb{U}^{\lfloor \frac{k}{2} \rfloor}$, *we have* $\nu^N_{k,v}(\mathsf{t}) = \mathrm{Ser}_N(|\mathsf{t}|_*, u^{\mathsf{t},v})$ *where*

$$|\mathsf{t}|_* := \sum_{2j-1 \leq k} t_{2j-1} \quad \text{and} \quad u^{\mathsf{t},v} := \sum_{2j \leq k} t_{2j} v_{2j} \delta_{\tau_j} \text{ where } \tau_1 := t_1 \text{ and } \tau_{j+1} = \tau_j + t_{j+1}.$$

thus $\log_N(\nu^N_{k,v}(\mathsf{t})) = Z_N(|\mathsf{t}|_*, X, u^{\mathsf{t},v})$ *(see Section 3) and*

$$|\mathsf{t}|_* + \|u^{\mathsf{t},v}\|_{\mathcal{U}} = |\mathsf{t}|. \tag{6.1}$$

Definition 6.3 (Good element of $\mathcal{G}^N(X)$). *A element $S \in \mathcal{G}^N(X)$ is good if there exists $k \in \mathbb{N}^*$, $v \in \mathbb{U}^{\lfloor \frac{k}{2} \rfloor}$ and $\mathfrak{t}^0 = (t_1^0, \dots, t_k^0) \in (0,\infty)^k$, such that, $\nu_{k,v}^N(\mathfrak{t}^0) = S$ and $d\nu_{k,v}^N(\mathfrak{t}^0)$ has rank equal to the dimension of $\mathcal{G}^N(X)$.*

Proposition 6.4. *There exists $T > 0$ such that $\exp_N(TX_0)$ is a good element of $\mathcal{G}^N(X)$. Moreover, Definition 6.3 holds with $k \leq N^N \dim(\mathrm{Lie}_{\mathbb{R}}^N(X_0, X_1, iX_1))$, and $|\mathfrak{t}^0|$ can be arbitrary small.*

Proof. We prove by induction on $\ell \in [\![1, N+1]\!]$ that there exists a good element S of $\mathcal{G}^N(X)$ of the form

$$S = \exp_N(TX_0 + Z) \text{ where } T > 0 \text{ and } Z \in \mathcal{L}_\ell^N(X) := \mathrm{span}\{b \in \mathrm{Br}(X); \ell \leq |b| \leq N\}, \qquad (6.2)$$

moreover Definition 6.3 holds with $k \leq N^{\ell-1} \dim(\mathrm{Lie}_{\mathbb{R}}^N(X_0, X_1, iX_1))$ and $|\mathfrak{t}^0|$ arbitrary small. This property for $\ell = N+1$ gives the conclusion.

Initialization for $\ell = 1$. Theorem 5.1 implies that a good element $S \in \mathcal{G}^N(X)$ exists. It is clearly of the form (6.2) with $\ell = 1$. It satisfies Definition 6.3 with $k = \dim(\mathrm{Lie}_{\mathbb{R}}^N(X_0, X_1, iX_1))$ and $|\mathfrak{t}^0|$ can be arbitrary small.

Heredity. Let $\ell \geq 1$. There exists a good element of S of $\mathcal{G}^N(X)$ of the form $S = \exp_N(TX_0 + Z)$ where $T > 0$ and $Z \in \mathcal{L}_\ell^N(X)$. We introduce the notation

$$Z = \sum_{j=1}^{N-1} Z_j \qquad \text{where} \qquad Z_j \in \mathrm{span}\{b \in \mathrm{Br}(X); \ell \leq |b| \leq N, n_1(b) = j\}.$$

For $p = 0, \dots, N-1$, we consider the unique morphism of algebra $\lambda_p : \mathcal{A}^N(X) \to \mathcal{A}^N(X)$ such that $\lambda_p(X_0) = X_0$ and $\lambda_p(X_1) = e^{i2\pi p/N} X_1$. Then

$$\lambda_p(S) = \exp_N(TX_0 + \lambda_p(Z)) \quad \text{where} \quad \lambda_p(Z) = \sum_{j=1}^{N-1} e^{i2\pi j \frac{p}{N}} Z_j.$$

If $S = \nu_{k,v}^N(\mathfrak{t})$ then, using the homogeneity properties of Item 4 of Proposition 3.4, $\lambda_p(S) = \nu_{k,v'}^N(\mathfrak{t})$ where $v' = e^{i2\pi p/N} v$. Thus $S\lambda_1(S) \dots \lambda_{N-1}(S)$ is a good element of $\mathcal{G}^N(X)$ of the form $\nu_{Nk,v'}^N(\mathfrak{t}')$ for some $\mathfrak{t}' \in (0,\infty)^{Nk}$ and $v' \in \mathbb{U}^{N\lfloor \frac{k}{2} \rfloor}$. By the Baker–Campbell–Hausdorff formula (Proposition 2.5),

$$S\lambda_1(S) \dots \lambda_{N-1}(S) = \exp_N(NTX_0 + Z_1 + Z_2)$$

where

$$Z_1 = \sum_{p=0}^{N} \lambda_p(Z) = \sum_{p=0}^{N} \sum_{j=1}^{N-1} e^{i2\pi j \frac{p}{N}} Z_j = \sum_{j=1}^{N-1} \left(\sum_{p=0}^{N-1} e^{i2\pi j \frac{p}{N}} \right) Z_j = 0$$

and Z_2 is a linear combination of iterated Lie brackets of X_0 and the $\lambda_j(Z)$ thus $Z_2 \in \mathcal{L}_{\ell+1}^N(X)$. $\square$

6.2 Arbitrary order splitting methods with complex coefficients

The goal of this section is to prove Theorem 1.11. By Proposition 4.6, it suffices to prove the following statement.

Theorem 6.5. *There exist $k \leq N^N \dim(\mathrm{Lie}_{\mathbb{R}}^N(X_0, X_1, iX_1))$ and coefficients $\alpha_1, \dots, \alpha_k > 0$ and $\beta_1, \dots, \beta_k \in \mathbb{C}$ such that the equality (5.2) holds in $\mathcal{G}^N(X)$. In other words, there exist a finite sum u of Dirac masses on $[0,1]$ with complex amplitudes such that $Z_N(1, X, u) = X_0 + X_1$, where Z_N is defined in Proposition 3.8.*

Proof of Theorem 6.5. Let $T > 0$, k, v given by Proposition 6.4. For $\eta > 0$ small enough, there exists $\mathfrak{t} \in (0,\infty)^k$ such that the following equality holds in $\mathcal{G}^N(X)$,

$$e^{TX_0+\eta X_1} = \nu_{k,v}^N(\mathfrak{t}) = \begin{cases} e^{t_1 X_0} e^{t_2 v_2 X_1} \dots e^{t_{2k-1} X_0} e^{t_k v_k X_1} & \text{if } k \text{ is even} \\ e^{t_1 X_0} e^{t_2 v_2 X_1} \dots e^{t_{2k-1} X_0} e^{t_k X_0} & \text{if } k \text{ is odd.} \end{cases}$$

By homogeneity, this implies (5.2) with $\alpha_j := t_{2j-1}/T > 0$ and $\beta_j := t_{2j} v_{2j}/\eta \in \mathbb{C}$. □

6.3 Control of complex systems

The goal of this section is to prove one implication of Theorem 1.12, precised Proposition 6.6, stating that the Lie algebra rank condition implies controllability. For the other implication (that controllability implies the Lie algebra rank condition), see e.g. [24, Theorem 3.17] for a modern reference. In particular, holomorphy is not required for Proposition 6.6 (but it is required for the other implication of Theorem 1.12).

We will use controls that are finite sums of Dirac masses (instead of L^1 functions, as in Definition 1.8), to emphasize the similarity with the splitting result above. However, a simple adaptation of the argument below, in line with Sussman's original argument, would allow us to conclude with piecewise constant controls, thus to get small-state STLC in the sense of Definition 1.8.

Proposition 6.6. *Let f_0, f_1 be smooth vector fields on a neighborhood of 0 in $\mathbb{C}^d$ with $f_0(0) = 0$ and $\mathbb{C}^d = \mathrm{Lie}_{\mathbb{C}}(f_0, f_1)(0)$. For every $\delta > 0$, there exists $r > 0$ such that, for every $x^* \in B_{\mathbb{R}^d}(0, r)$, there exist $T \in [0, \delta]$, $u \in \mathcal{U}$ with $\|u\|_{\mathcal{U}} < \delta$ such that $x(T; f, u, 0) = x^*$.*

Proof. Let $\delta > 0$ and $X = \{X_0, X_1\}$. We have $\mathrm{Lie}_{\mathbb{C}}(f_0, f_1) = i\mathbb{R}f_0 + \mathrm{Lie}_{\mathbb{R}}(f_0, f_1, if_1)$ and $f_0(0) = 0$ thus $\mathrm{Lie}_{\mathbb{R}}(f_0, f_1, if_1)(0) = \mathbb{C}^d$. Therefore, there exist $b_1, \dots, b_{2d} \in \mathrm{Br}(X) \setminus \{X_0\}$ such that $(f_{b_1}(0), \dots, f_{b_{2d}}(0))$ is an $\mathbb{R}$-basis of $\mathbb{C}^d$. Let $N := \max\{|b_j| + n_1(b_j); j \in [\![1, 2d]\!]\}$. Let C, η given by Proposition 4.7.

Let $T \in (0, \delta)$, $k \leq N^N \dim(\mathrm{Lie}_{\mathbb{R}}^N(X_0, X_1, iX_1))$, $v \in \mathbb{U}^{\lfloor \frac{k}{2} \rfloor}$, $\mathfrak{t}^0 \in (0,\infty)^k$ be given by Proposition 6.4. By the inverse mapping theorem, there exist an open neighborhood Ω of TX_0 in $\mathcal{L}^N(X)$, an open neighborhood $\mathfrak{T}$ of $\mathfrak{t}^0$ in $(0,\infty)^k$ and a C^1 map $\psi : \Omega \to \mathfrak{T}$ such that, for every $Z \in \Omega$, $\mathfrak{t} := \psi(Z)$ satisfies $Z_N(|\mathfrak{t}|_*, X, u^{\mathfrak{t},v}) = Z$. One may assume Ω and $\mathfrak{T}$ small enough so that $|\psi(Z)| \leq 2|\mathfrak{t}^0|$ for every $Z \in \Omega$.

There exists $\rho^* > 0$ such that, for every $\rho \in \overline{B}_{\mathbb{R}^{2d}}(0, \rho^*)$, $\sum_{j=1}^{2d} \rho_j b_j \in \Omega$ and then $\mathfrak{t}_\rho := \psi(\sum_{j=1}^{2d} \rho_j b_j)$ satisfies $|\mathfrak{t}_\rho| \leq 2|\mathfrak{t}^0|$ and $Z_N(|\mathfrak{t}_\rho|_*, X, u^{\mathfrak{t}_\rho, v}) = \sum_{j=1}^{2d} \rho_j b_j$.

For $\rho \in \overline{B}_{\mathbb{R}^{2d}}(0, \rho^*)$ and $\varepsilon > 0$ we define

$$\begin{array}{rccl} u_\varepsilon^{\mathfrak{t}_\rho} : & (0, \varepsilon|\mathfrak{t}_\rho|_*) & \to & \mathbb{C} \\ & t & \mapsto & \varepsilon u^{\mathfrak{t}_\rho}\left(\frac{t}{\varepsilon}\right). \end{array}$$

Then, by (6.1)

$$\varepsilon|\mathfrak{t}_\rho|_* + \|u_\varepsilon^{\mathfrak{t}_\rho}\|_{\mathcal{U}} = \varepsilon|\mathfrak{t}_\rho| \leq 2|\mathfrak{t}^0|\varepsilon \tag{6.3}$$

Moreover, the homogeneity properties of Item 4 of Proposition 3.4 of the coordinates of the first kind proves

$$Z_N(\varepsilon|\mathfrak{t}_\rho|_*, X, u_\varepsilon^{\mathfrak{t}_\rho}) = \sum_{j=1}^{2d} \varepsilon^{|b_j| + n_1(b_j)} \rho_j b_j. \tag{6.4}$$

We introduce α, C', r as in (5.4) and (5.5), with sums indexed by $j \in [\![1, 2d]\!]$ instead of $j \in [\![1, d]\!]$. For $z = |z| \sum_{j=1}^{2d} \omega_j f_{b_j}(0) \in B_{\mathbb{C}^d}(0, r)$, we define

$$\varepsilon(z) := |z|^\alpha \quad \text{and} \quad \rho(z) := \sum_{j=1}^{2d} |z|^{1-\alpha(|b_j|+n_1(b_j))} \omega_j f_{b_j}(0)$$

(for $z = 0$, we take $\rho(z) = 0$ so that ρ be continuous). Then, by (5.4) and (5.5)

$$|\rho(z)| \leq |z|^{1-\alpha N} \sum_{j=1}^{2d} |\omega_j||f_{b_j}(0)| \leq r^{1-\alpha N} C' \leq \rho^*$$

thus the control $u_{\varepsilon(z)}^{\mathfrak{t}_{\rho(z)}}$ is well defined. To simplify the notation we write T_z instead of $\varepsilon(z)|\mathfrak{t}_{\rho(z)}|$ and u_z instead of $u_{\varepsilon(z)}^{\mathfrak{t}_{\rho(z)}}$. Then, using (6.3), (5.5) and (6.4), we obtain

$$T_z + \|u_z\|_{\mathcal{U}} \leq 2|\mathfrak{t}^0||z|^\alpha \leq \delta \quad \text{and} \quad Z_N(T_z, f, u_z)(0) = z.$$

Thus, by Proposition 4.3,

$$\begin{aligned} |x(T_z; f, u_z, 0) - z| &\leq C\left((T_z + \|u_z\|_{\mathcal{U}})^{N+1} + (T_z + \|u_z\|_{\mathcal{U}})|x(T_z, f, u_z, 0)|\right) \\ &\leq C\left((2|\mathfrak{t}^0||z|^\alpha)^{N+1} + (2|\mathfrak{t}^0||z|^\alpha)|x(T_z, f, u_z, 0)|\right). \end{aligned}$$

A Brouwer fixed point argument ends the proof as for Proposition 5.7. □

7 Order restrictions for signed real-valued methods

In this section, we are interested in order restrictions for $(\mathbb{R}^+, \mathbb{R})$ splitting methods, i.e. with positive coefficients along f_0 and real-valued coefficients along f_1 and other additional commutator flows. In the spirit of [6] for control theory, we exhibit a list of obstructions to the construction of such splitting methods, which are linked with quadratic quantities with respect to the control (or coefficients along f_1).

We start with the first and second obstructions corresponding to Theorems 1.13 and 1.18. We will see in Section 7.3 that they are consequences of the more general result Theorem 7.3.

To prove these results (and the general case), we interpret such splitting methods (with positive coefficients along X_0 and real-valued coefficients along X_1 and the other commutators) as trajectories of the control-affine formal system

$$\dot{S}(t) = S(t)\left(X_0 + u_{X_1}(t)X_1 + u_{W_1}(t)W_1 + \cdots\right) \tag{7.1}$$

for a finite sum of $m \geq 1$ controls along m brackets. Up to a rescaling, we can work on $t \in [0, 1]$. Then, the splitting method is of order N if and only if $\pi_N(\log S(1)) = Z_N(1, u) = X_0 + X_1$.

The final state $S(1) = \exp(X_0 + X_1)$ is achieved by the reference control $\bar{u} := (1, 0, \ldots, 0)$, which corresponds to $u_{X_1} \equiv 1$ and $u_b \equiv 0$ for $b \neq X_1$.

Our proofs rely on the coordinates of the second kind. Thanks to Proposition 3.14, the splitting method is of order N if and only if $\xi_b(t, u) = \xi_b(t, \bar{u})$ for all $b \in \mathcal{B}_\star^N$, if and only if $\zeta_b(t, u) = \zeta_b(t, \bar{u})$ for all $b \in \mathcal{B}_\star^N$. Moreover, $\zeta_b(1, \bar{u}) = 1$ if $b \in \{X_0, X_1\}$ and 0 otherwise.

7.1 The first obstruction

In this setting, we have a single scalar control $u = u_{X_1}$ and we are looking at the formal system $\dot{S} = S(X_0 + uX_1)$. Recall from Definition 1.2 that $M_1 = (X_1, X_0)$ and $M_2 = (M_1, X_0)$. The main obstruction of length 3 is associated with the Lie bracket $W_1 = \mathrm{ad}_{X_1}^2(X_0)$. Theorem 1.13 is a direct consequence of the following key positivity argument. Recall that $\bar{u}$ denotes the constant control $\bar{u}(t) \equiv 1$.

Proposition 7.1. *Let $u = u_{X_1} : [0, 1] \to \mathbb{R}$ be a finite sum of Dirac masses such that*

$$\zeta_{X_1}(1, u) = \zeta_{X_1}(1, \bar{u}) \quad \textit{and} \quad \zeta_{M_1}(1, u) = \zeta_{M_1}(1, \bar{u}). \tag{7.2}$$

Then

$$\zeta_{W_1}(1,u) + \zeta_{M_2}(1,u) > 0. \tag{7.3}$$

In particular, if $\zeta_{M_2}(1,u) = \zeta_{M_2}(1,\bar{u}) = 0$*, then*

$$\zeta_{W_1}(1,u) > 0. \tag{7.4}$$

Proof. By definition, one has $\xi_{X_1}(t,u) = \int_0^t u =: U(t)$ and $\xi_{M_1}(t,u) = \int_0^t U$. Let us write $u = 1+v$ so that we have $\xi_{X_1}(t,u) = U(t) = t + V(t)$ where $V(t) := \int_0^t v$. By the induction formula

$$\begin{aligned}
\xi_{W_1}(t,u) &= \frac{1}{2}\int_0^t (\xi_{X_1}(s,u))^2 \, \mathrm{d}s \\
&= \frac{1}{2}\int_0^t (s + V(s))^2 \, \mathrm{d}s \\
&= \frac{1}{2}\int_0^t (V(s))^2 \, \mathrm{d}s + \int_0^t sV(s) \, \mathrm{d}s + \frac{1}{2}\int_0^t s^2 \, \mathrm{d}s.
\end{aligned} \tag{7.5}$$

Recalling that $\bar{u} = 1$, one has $\xi_{X_1}(s,\bar{u}) = s$ on $(0,1)$ and thus, at time 1, we obtain

$$\xi_{W_1}(1,u) = \frac{1}{2}\int_0^1 (V(s))^2 \, \mathrm{d}s + (\xi_{M_1}(1,u) - \xi_{M_1}(1,\bar{u})) - (\xi_{M_2}(1,u) - \xi_{M_2}(1,\bar{u})) + \xi_{W_1}(1,\bar{u}). \tag{7.6}$$

If u satisfies (7.2), then, by (3.24) (or, more concretely, by (3.25)), $\zeta_{M_2}(1,u) = \xi_{M_2}(1,u) - \xi_{M_2}(1,\bar{u})$ and $\zeta_{W_1}(1,u) = \xi_{W_1}(1,u) - \xi_{W_1}(1,\bar{u})$. Hence,

$$\zeta_{W_1}(1,u) + \zeta_{M_2}(1,u) = \frac{1}{2}\int_0^1 (V(t))^2 \, \mathrm{d}t = \frac{1}{2}\int_0^1 (\xi_{X_1}(t,u) - t)^2 \, \mathrm{d}t \tag{7.7}$$

Heuristically, the positive quantity is the square of the H^{-1} norm of $u - 1$ (i.e. the L^2 norm of its primitive), which is well-defined when u is a sum of Dirac masses, and strictly positive since one cannot have $u \equiv 1$. □

7.2 The second obstruction

In this setting we have two scalar controls u_{X_1} and u_{W_1} and we are looking at the formal system $\dot{S} = S(X_0 + u_{X_1}X_1 + u_{W_1}W_1)$. The second control u_{W_1} was added to the scheme to circumvent the order restriction of the first obstruction. For the reference control $\bar{u} = (1,0)$, we check that

$$\xi_{X_1}(t,\bar{u}) = t, \quad \xi_{M_1}(t,\bar{u}) = \frac{t^2}{2}, \quad \xi_{M_2}(t,\bar{u}) = \frac{t^3}{6}, \quad \xi_{W_2}(t,\bar{u}) = \frac{t^5}{40}. \tag{7.8}$$

Working as above, we prove the following result, which implies Theorem 1.18.

Proposition 7.2. *Let* $u = (u_{X_1}, u_{W_1}) : [0,1] \to \mathbb{R}^2$ *be a finite sum of Dirac masses such that*

$$\zeta_{M_\nu}(1,u) = \zeta_{M_\nu}(1,\bar{u}) \quad \textit{for} \quad \nu \in \{0,1,2,3\}. \tag{7.9}$$

Then

$$\zeta_{W_2}(1,u) + \zeta_{M_4}(1,u) > 0. \tag{7.10}$$

In particular, if $\zeta_{M_4}(1,u) = \zeta_{M_4}(1,\bar{u}) = 0$*, then*

$$\zeta_{W_2}(1,u) > 0. \tag{7.11}$$

Proof. The proof follows the same lines as the one of Proposition 7.1. We obtain more precisely that, under the assumption (7.9),

$$\zeta_{W_2}(1,u) + \zeta_{M_4}(1,u) = \frac{1}{2}\int_0^1 \left(\xi_{M_1}(t,u) - \frac{t^2}{2}\right)^2 \mathrm{d}t, \tag{7.12}$$

which corresponds to the square of the H^{-2} Sobolev norm of $u_{X_1} - 1$ (the L^2 norm of its second primitive) which as above, is well-defined and strictly positive. □

7.3 A general result

The particular cases of Theorems 1.13 and 1.18 admit the following generalization.

Theorem 7.3. *Let $N \geq 1$. Consider an $(\mathbb{R}^+, \mathbb{R})$ splitting method, involving X_0 (with positive coefficients) and a set $\mathcal{C} \subset \mathcal{B}_\star \setminus \{X_0\}$ of flows (with real-valued coefficients), say with $X_1 \in \mathcal{C}$. Assume that $\mathcal{C} \cap \{M_N, \dots, M_{2N}, W_N\} = \emptyset$. Then the method is of order at most $2N$.*

In the spirit of the previous proofs, this result is a consequence of the following proposition.

Proposition 7.4. *Under the assumptions of Theorem 7.3, let $u \in \mathcal{U}$ be a multi-control such that, for all $b \in \{X_1, M_1, \dots, M_{2N}\}$, $\zeta_b(1,u) = \zeta_b(1,\bar{u})$, where $\bar{u} = (1, 0, \dots, 0)$. Then, $\zeta_{W_N}(1,u) > 0$.*

Proof. Let us denote by $\bar{u} := (1, 0, \dots, 0)$ the reference control corresponding to $u_{X_1} \equiv 1$ and $u_b \equiv 0$ for $b \neq X_1$. One checks by induction that there exist constants $\gamma_b > 0$ such that, for all $b \in \mathcal{B}$ and $t \geq 0$, $\xi_b(t, \bar{u}) = \gamma_b t^{|b|}$. For another control u, we then write

$$\xi_b(t,u) = \gamma_b t^{|b|} + \widetilde{\xi}_b(t,u). \tag{7.13}$$

Let $\gamma_N := \gamma_{M_{N-1}}$. Since $W_N \notin \mathcal{C}$ (the set of controlled flows), one checks from the induction formula for the coordinates of the second kind that

$$\begin{aligned}
\xi_{W_N}(t,u) &= \frac{1}{2}\int_0^t \xi_{M_{N-1}}(s,u)^2 \,\mathrm{d}s \\
&= \frac{1}{2}\int_0^t \left(\gamma_N s^N + \widetilde{\xi}_{M_{N-1}}(s,u)\right)^2 \,\mathrm{d}s \\
&= \frac{1}{2}\int_0^t \widetilde{\xi}_{M_{N-1}}(s,u)^2 \,\mathrm{d}s + \gamma_N \int_0^t s^N \xi_{M_{N-1}}(s,u)\,\mathrm{d}s - \frac{1}{2}\gamma_N^2 \int_0^t s^{2N}\,\mathrm{d}s.
\end{aligned} \tag{7.14}$$

Since $\mathcal{C} \cap \{M_N, \dots, M_{2N}\} = \emptyset$, one checks that $\xi_{M_\nu}(t,u) = \int_0^t \xi_{M_{\nu-1}}$ for $\nu \in \{N, \dots, 2N\}$. Moreover, if u is such that $\xi_{M_\nu}(1,u) = \xi_{M_\nu}(1,\bar{u})$ for $0 \leq \nu \leq 2N$, then one checks using integrations by part that

$$\xi_{W_N}(1,u) = \frac{1}{2}\int_0^1 \widetilde{\xi}_{M_{N-1}}(s,u)^2\,\mathrm{d}s + \xi_{W_N}(1,\bar{u}). \tag{7.15}$$

Using Proposition 3.14, and the assumption that u is such that $\xi_{M_\nu}(1,u) = \xi_{M_\nu}(1,\bar{u})$ for $0 \leq \nu \leq 2N$, one can write

$$\zeta_{W_N}(1,u) = \frac{1}{2}\int_0^1 \widetilde{\xi}_{M_{N-1}}(s,u)^2\,\mathrm{d}s. \tag{7.16}$$

As previously, this quantity corresponds to the (squared) H^{-N} Sobolev norm of $u_{X_1} - 1$ (the L^2 norm of its N-th primitive), which, as above, is well-defined and strictly positive. $\square$

8 High-order methods using commutator flows

To complete the picture, we prove the following positive counterpart to Theorems 1.13 and 1.18 and more generally to Theorem 7.3. In particular, it encompasses Theorems 1.16 and 1.19.

Theorem 8.1. *There exist $(\mathbb{R}^+, \mathbb{R})$ splitting methods of order*

- *2 involving only X_0 and X_1,*
- *4 involving only X_0 and X_1, W_1,*
- *6 involving only X_0 and X_1, W_1, W_2,*

- *8 involving only X_0 and $X_1, W_1, W_2, W_3, \mathrm{ad}^2_{W_1}(X_0)$.*

Using the interpretation of splitting methods as trajectories of control-affine formal systems of the form (7.1), and Proposition 3.14, finding a splitting method of order N is equivalent to finding a control u which is a sum of disjoint Dirac masses such that $Z_N(1,u) = X_0 + X_1$, or, equivalently, such that $\xi_b(1,u) = \xi_b(1,\bar{u})$ for all $b \in \mathcal{B}^N_\star$, where we recall that $\bar{u} = (1, 0, \dots, 0)$. Since $\xi_{X_0}(1,u) = 1$ for any u, and using time and control homogeneity (see Item 4 of Proposition 3.4), it suffices to prove that the finite-dimensional system $x = \{\xi_b; b \in \mathcal{B}^N_\star, b \neq X_0\}$ is small-time locally controllable.

We will rely on the following well-known control result[2], due to Sussman [58].

Proposition 8.2. *Let $m \geq 1$ and $f_0, f_1, \dots, f_m \in \mathcal{C}^\omega(\mathbb{R}^d;\mathbb{R}^d)$ with $f_0(0)$. Let $\omega_0, \omega_, \dots, \omega_m \geq 0$. For $b \in \mathrm{Br}(\{X_0, X_1, \dots, X_m\})$, define the* weight of b *as*

$$\omega(b) := \omega_0 n_0(b) + \cdots + \omega_m n_m(b), \tag{8.1}$$

where $n_i(b)$ denotes the number of X_i in b. Assume that there exists $\bar{\omega} \geq 0$ such that

1. $\mathrm{span}\{f_b(0); \omega(b) \leq \bar{\omega}\} = \mathbb{R}^d$,
2. *for all $b \in \mathrm{Br}(\{X_0, X_1, \dots, X_m\})$ with $\omega(b) \leq \bar{\omega}$, $n_0(b)$ odd and $n_1(b), \dots, n_m(b)$ even,*

$$f_b(0) \in \mathrm{span}\{f_c(0); \omega(c) < \omega(b)\}. \tag{8.2}$$

Then the system $\dot{x} = f_0(x) + u_1 f_1(x) + \cdots + u_m f_m(x)$ is small-time locally controllable with controls which are sum of disjoint Dirac masses.

8.1 Method of order 2

One has $\mathcal{B}^2_\star = \{X_0, X_1, M_1\}$, where $M_1 = (X_1, X_0)$. We use a single scalar control, along X_1. We consider the system $x = (\xi_{X_1}, \xi_{M_1})$ on $\mathbb{R}^2$, whose dynamic is given by

$$\begin{cases} \dot{\xi}_{X_1} = u \\ \dot{\xi}_{M_1} = \xi_{X_1} \end{cases} \tag{8.3}$$

Writing $x = (x_1, x_2)$, this corresponds to $f_1(x) = (1, 0)$ and $f_0(x) = (0, x_1)$. In particular, all brackets involving f_1 at least twice vanish identically. So there is no bracket to compensate and Item 2 of Proposition 8.2 is verified. Moreover, $f_{X_1}(0) = f_1(0) = (1, 0)$ and $f_{M_1}(0) = [f_1, f_0](0) = (0, 1)$. Hence Item 1 of Proposition 8.2 is verified for example with $\omega_0 = \omega_1 = 1$ and $\bar{\omega} = 2$.

8.2 Method of order 4

One has $\mathcal{B}^4_\star = \{X_0, X_1, M_1, M_2, M_3, W_1, (W_1, X_0), \mathrm{ad}^3_{X_1}(X_0)\}$. We use two scalar controls, along X_1 and W_1. Writing the system for $x = \{\xi_b; b \in \mathcal{B}^4_\star \setminus \{X_0\}\}$, set on $\mathbb{R}^7$, and using the order given previously, we have

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2 \\ \dot{x}_4 = x_3 \\ \dot{x}_5 = \frac{1}{2} x_1^2 + u_2 \\ \dot{x}_6 = x_5 \\ \dot{x}_7 = \frac{1}{3!} x_1^3 + x_1 u_2 \end{cases} \tag{8.4}$$

[2] We use here a slightly different version than Sussman's one, which can be obtained by adapting his argument in the spirit of Sections 5 and 6.

which corresponds to $f_1(x) = (1, 0, 0, 0, 0, 0, 0)$, $f_0(x) = (0, x_1, x_2, x_3, x_1^2/2, x_5, x_1^3/6)$ and $f_2(x) = (0, 0, 0, 0, 1, 0, x_1)$. One checks that $f_2(x) = f_{W_1}(x)$ (which is $[f_1, [f_1, f_0]](x)$ by definition).

By construction $f_{b_j}(0) = e_j$ for $1 \le j \le 7$ where the b_j are the ordered elements of $\mathcal{B}^4_\star \setminus \{X_0\}$.

We plan to apply Proposition 8.2 with $\omega_0 = \omega_1 = 1$ and $\omega_2 = 3 - \varepsilon$. In particular, by the previous remark, Item 1 is verified with $\bar{\omega} = 4$.

Let $b \in \mathrm{Br}(\{X_0, X_1, X_2\})$ with $n_0(b)$ odd, $n_1(b)$ even and $n_2(b)$ even, and $\omega(b) \le 4$. Hence $\omega_2 n_2(b) \le 4$. If $\varepsilon < 1$, this implies that $n_2(b) = 0$. Hence the only possibility is that $n_1(b) = 2$ and $n_0(b) = 1$, so that $\omega(b) = 3$ and $f_b(0) = \pm f_{W_1}(0)$. Hence, $f_b(0)$ is compensated by $f_2(0)$, corresponding to the bracket $c = X_2$, with $\omega(X_2) = 3 - \varepsilon$.

8.3 Method of order 6

Here $\mathcal{B}^6_\star$ is of cardinal $22 + 1$ (for X_0). We use three scalar controls, along X_1, W_1 and W_2. As above, we can write the system for $x = \{\xi_b; b \in \mathcal{B}^6_\star \setminus \{X_0\}\}$, set on $\mathbb{R}^{22}$. We have $f_1 = (1, 0, \dots, 0)$, $f_2 = f_{W_1}$ and $f_3 = f_{W_2}$.

We plan to apply Proposition 8.2 with $\omega_0 = \omega_1 = 1$, $\omega_2 = 3 - \varepsilon_2$ and $\omega_3 = 5 - \varepsilon_3$.

Condition Item 1 is verified with $\bar{\omega} = 6$. We now verify the compensation condition Item 2. Let $b \in \mathrm{Br}(\{X_0, X_1, X_2, X_3\})$ with $n_0(b)$ odd, $n_1(b), n_2(b), n_3(b)$ even, and $\omega(b) \le 6$. In particular $n_0(b) \ge 1$.

- If $1 + 2(5 - \varepsilon_3) > 6$, this implies that $n_3(b) = 0$. This holds for $\varepsilon_3 < 5/2$.
- If $1 + 2(3 - \varepsilon_2) > 6$, this implies that $n_2(b) = 0$. This holds for $\varepsilon_2 < 1/2$.
- Thus we can assume that $n_2(b) = n_3(b) = 0$, and $n_1(b)$ is either 2 or 4. Since $b \in \mathrm{Br}(\{X_0, X_1\})$, by linearity, it suffices to check the compensation for $b \in \mathcal{B}_\star$. Moreover, $\omega(b) = |b|$ so $b \in \mathcal{B}^6_\star$.
  - Case $n_1(b) = 2$. Moreover $n_0(b)$ is odd so is 1 or 3. Hence, $b \in \mathcal{B}^5_\star$. By (2.15), $b \in \{W_1, ((W_1, X_0), X_0), W_2\}$.
    - If $b = W_1$, $\omega(b) = 3$ and $f_b(0) = f_{W_1}(0) = f_2(0) = f_{X_2}(0)$ with $\omega(X_2) = 3 - \varepsilon_2$. So the compensation holds iff $\varepsilon_2 > 0$.
    - If $b = ((W_1, X_0), X_0)$, $\omega(b) = 5$ and $f_b(0) = [[f_{W_1}, f_0], f_0](0) = [[f_2, f_0], f_0](0) = f_{((X_2, X_0), X_0)}$ with $\omega(((X_2, X_0), X_0)) = \omega(X_2) + 2 = 5 - \varepsilon_2$. So the compensation holds iff $\varepsilon_2 > 0$.
    - If $b = W_2$, $\omega(b) = 5$ and $f_b(0) = f_{W_2}(0) = f_3(0) = f_{X_3}(0)$ with $\omega(X_3) = 5 - \varepsilon_3$. So the compensation holds iff $\varepsilon_3 > 0$.
  - Case $n_1(b) = 4$. Since $n_0(b)$ is odd, $n_0(b) = 1$ and $b = \mathrm{ad}^4_{X_1}(X_0)$ with $\omega(b) = 5$. Since $\mathrm{ad}^4_{X_1}(X_0) = \mathrm{ad}^2_{X_1}(W_1)$, $f_b(0) = f_{\mathrm{ad}^2_{X_1}(X_2)}(0)$ and $\omega(\mathrm{ad}^2_{X_1}(X_2)) = 2 + \omega_2$. So the compensation holds if and only if $\varepsilon_2 > 0$.

In summary, Sussmann's result applies provided that $\varepsilon_2 \in (0, 1/2)$ and $\varepsilon_3 \in (0, 5/2)$.

8.4 Method of order 8

Here $\mathcal{B}^8_\star$ is of cardinal 70+1 (for X_0). We use five scalar controls, along X_1, W_1, W_2, W_3 and $Q_1^\flat := \mathrm{ad}^2_{W_1}(X_0)$. We have $f_2 = f_{W_1}$, $f_3 = f_{W_2}$, $f_4 = f_{W_3}$ and $f_5 = f_{Q_1^\flat}$. We plan to apply Proposition 8.2 with $\omega_0 = \omega_1 = 1$, $\omega_2 = 3 - \varepsilon_2$, $\omega_3 = 5 - \varepsilon_3$, $\omega_4 = 7 - \varepsilon_4$ and $\omega_5 = 7 - \varepsilon_5$.

Condition Item 1 is verified with $\bar{\omega} = 8$. We now verify the compensation condition Item 2. Let $b \in \mathrm{Br}(\{X_0, X_1, \dots, X_5\})$ with $n_0(b)$ odd, $n_1(b), \dots, n_5(b)$ even, and $\omega(b) \le 8$. In particular $n_0(b) \ge 1$.

Working as above, we obtain that $n_3(b) = n_4(b) = n_5(b) = 0$ and $n_2(b) \in \{0, 2\}$ when the ε_i are small enough, for example $\varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5 \le 1$ suffices to ensure this property.

- Case $n_2(b) = 2$. If $\varepsilon_2 < 1/2$, this implies that $n_1(b) = 0$ and $n_0(b) = 1$. Hence, up to a sign, b is of the form $\mathrm{ad}^2_{X_2}(X_0)$. Since $f_2 = f_{W_1}$ and $Q_1^\flat = \mathrm{ad}^2_{W_1}(X_0)$, we have $f_b(0) = \pm f_5(0)$. Since $\omega(b) = 1 + 2(3 - \varepsilon_2)$ and $\omega(X_5) = 7 - \varepsilon_5$, the compensation holds if and only if $2\varepsilon_2 < \varepsilon_5$.
- Case $n_2(b) = 0$. As in the previous method of order 6, since $b \in \mathrm{Br}(\{X_0, X_1\})$, by linearity, it suffices to check the compensation for $b \in \mathcal{B}_\star$. Moreover, $\omega(b) = |b|$ so $b \in \mathcal{B}^8_\star$. Moreover, $n_1(b) \in \{2, 4, 6\}$.
  - Case $n_1(b) = 2$. Then $b = W_j$ for $j \in \{1, 2, 3\}$ (or of the form $((W_j, X_0), \dots, X_0))$. As in the previous paragraphs, the compensation holds provided that $\varepsilon_2, \varepsilon_3, \varepsilon_4 > 0$.
  - Case $n_1(b) = 4$. Then $n_0(b) \in \{1, 3\}$. Using the explicit description of such elements of $\mathcal{B}^7_\star$ (see [7, Equation (1.11)]), we know that b is of one of the following forms:
    - $b = Q_1$, with $\omega(b) = 5$, so $f_b(0) = \mathrm{ad}^4_{f_1}(f_0)(0) = \mathrm{ad}^2_{f_1}(f_2) = f_{(X_1,(X_1,X_2))}(0)$ with $\omega((X_1, (X_1, X_2))) = 5 - \varepsilon_2$, requiring $\varepsilon_2 > 0$.
    - $b = Q_1^\flat$, with $\omega(b) = 7$, so $f_b(0) = f_5(0)$, where $\omega(X_5) = 7 - \varepsilon_5$, requiring $\varepsilon_5 > 0$,
    - $b = (M_2, (X_1, W_1))$, with $\omega(b) = 7$, so $f_b(0) = [f_{M_2}, [f_1, f_2]](0) = f_{(M_2,(X_1,X_2))}(0)$ where $\omega((M_2, (X_1, X_2))) = 7 - \varepsilon_2$, requiring $\varepsilon_2 > 0$.
    - $b = ((Q_1, X_0), X_0)$, with $\omega(b) = 7$, so $f_b(0) = f_c(0)$ where $c = ((\mathrm{ad}^2_{X_1}(X_2), X_0), X_0)$ with $\omega(c) = 7 - \varepsilon_2$, requiring $\varepsilon_2 > 0$.
  - Case $n_1(b) = 6$. Then $b = \mathrm{ad}^6_{X_1}(X_0) = \mathrm{ad}^4_{X_1}(W_1)$, with $\omega(b) = 7$. Thus $f_b(0) = \mathrm{ad}^4_{f_1}(f_2)(0) = f_{\mathrm{ad}^4_{X_1}(X_2)}(0)$ where $\omega(\mathrm{ad}^4_{X_1}(X_2)) = 7 - \varepsilon_2$, requiring $\varepsilon_2 > 0$.

In summary, Sussmann's result applies provided that $\varepsilon_i \in (0, 1]$ with $2\varepsilon_2 < \varepsilon_5$.

9 High-order methods relying on degeneracies

In this section, we prove results of the following form: we consider fixed vector fields f_0, f_1 and we assume that there exists an $(\mathbb{R}^+, \mathbb{R})$ splitting method which achieves (for these specific vector fields) a better order than the maximal possible "universal" order of Definition 1.5 which is required to be independent of (f_0, f_1). We prove that this entails vectorial relations between some commutators f_b for $b \in \mathrm{Br}(X)$ of f_0 and f_1. We will use the following lemma.

Lemma 9.1. *Let $N \geq 1$ and $y_1, \dots, y_N$ and $z_1, \dots, z_N$ be smooth vector fields on $\mathbb{R}^d$. For $T \geq 0$, define $y(T) := Ty_1 + T^2 y_2 + \dots + T^N y_N$ and $z(T) := Tz_1 + \dots + T^N z_N$. Assume that*

$$\exp(y(T)) - \exp(z(T)) = \underset{T\to 0}{O}(T^{N+1}). \tag{9.1}$$

Then, for all $1 \leq i \leq N$, $y_i = z_i$.

Proof. Using the BCH formula, we have

$$\exp(-z(T)) \exp(y(T)) - \exp(\mathrm{BCH}_N(-z(T), y(T))) = \underset{T\to 0}{O}(T^{N+1}). \tag{9.2}$$

Hence, letting $g(T) := \mathrm{BCH}_N(-z(T), y(T))$, we obtain from (9.1) that

$$\exp(g(T)) - \mathrm{Id} = \underset{T\to 0}{O}(T^{N+1}). \tag{9.3}$$

Moreover, for any given vector field h on $\mathbb{R}^d$, Grönwall's lemma proves that, for every $x_0 \in \mathbb{R}^d$,

$$|e^h x_0 - x_0 - h(x_0)| \leq |h(x_0)| \|h\|_{\mathcal{C}^1} e^{\|h\|_{\mathcal{C}^1}}, \tag{9.4}$$

where the $\mathcal{C}^1$-norm is relative to a compact neighborhood of x_0. Moreover, since $y(T) = O(T)$ and $z(T) = O(T)$, $g(T) = O(T)$. Combining these arguments leads to the fact that $g(T) = O(T^2)$ and then by induction that $g(T) = O(T^{N+1})$.

From this estimate, one proves by induction that $y_i = z_i$ for $1 \leq i \leq N$. □

9.1 This first bad Lie bracket

We prove Theorem 1.15, using the coercivity argument of Proposition 7.1.

Proof of Theorem 1.15. With the notations of Definition 1.2, (2.15) implies that

$$\mathcal{B}^3_\star = \{X_0, X_1, M_1, M_2, W_1\}. \tag{9.5}$$

Let f_0, f_1 be smooth vector fields on $\mathbb{R}^d$. If $f_1 = 0$ or $f_{M_1} = 0$ then $f_{W_1} = 0$ and the conclusion holds. Thus, we can assume that $f_1 \neq 0$ and $f_{M_1} \neq 0$.

We assume that there exists an $(\mathbb{R}^+, \mathbb{R})$ splitting method of order 3 relative to (f_0, f_1). Let $u \in \mathcal{U}$ be the associated control. To lighten the notations, since u is fixed, we write ζ_b instead of $\zeta_b(1, u)$.

Using the assumption and Magnus estimate, we obtain

$$e^{Z_3(1,Tf,u)} - e^{T(f_0+f_1)} = \underset{T\to 0}{O}(T^4). \tag{9.6}$$

By Lemma 9.1, this implies that $(1-1)f_0 = 0$, $(\zeta_{X_1} - 1)f_1 = 0$, $(\zeta_{M_1} - 0)f_{M_2} = 0$ and

$$\zeta_{M_2} f_{M_2} + \zeta_{W_1} f_{W_1} = 0. \tag{9.7}$$

Since $f_1 \neq 0$ and $f_{M_1} \neq 0$, we have $\zeta_{X_1} = 1$ and $\zeta_{M_1} = 0$. By Proposition 7.1, $\zeta_{W_1} + \zeta_{M_2} > 0$, thus both coefficients cannot be simultaneously null and f_{W_1} and f_{M_2} are linearly dependent. □

Remark 9.2. *We proved above that the existence of an* $(\mathbb{R}^+, \mathbb{R})$ *splitting method relative to* (f_0, f_1) *implies that* f_{W_1} *and* f_{M_2} *are linearly dependent. In control theory, one is used to the conclusion that* f_{W_1} *is in the span of the* f_{M_ν} *(see Theorem 1.14). Here, however, it is not true that one can conclude that there exists* $\lambda \in \mathbb{R}$ *such that* $f_{W_1} = \lambda f_{M_2}$.

Consider the control on $[0,1]$ *given by*

$$u := \frac{1}{3}\delta_{t=0} + \frac{2}{3}\delta_{t=\frac{3}{4}}. \tag{9.8}$$

This corresponds to $(\mathbb{R}^+, \mathbb{R})$ *(even* $(\mathbb{R}^+, \mathbb{R}^+)$*) splitting method*

$$e^{\frac{1}{4}Tf_0} e^{\frac{2}{3}Tf_1} e^{\frac{3}{4}Tf_0} e^{\frac{1}{3}Tf_1}. \tag{9.9}$$

For this control, the primitive of u *is* $U(t) = \frac{1}{3}$ *for* $t \in [0, \frac{3}{4})$ *and* $U(t) = 1$ *for* $t \in [\frac{3}{4}, 1]$. *In particular,* $U(1) = 1$, $\int_0^1 U = \frac{1}{2}$ *and* $\frac{1}{2}\int_0^1 U^2 = \frac{1}{6}$. *Thus, denoting by* $\bar{u} \equiv 1$ *the constant reference control, we have* $\zeta_{X_1}(1,u) = \zeta_{X_1}(1,\bar{u})$, $\zeta_{M_1}(1,u) = \zeta_{M_1}(1,\bar{u})$ *and* $\zeta_{W_1}(1,u) = \zeta_{W_1}(1,\bar{u})$.

Hence, if the vector fields (f_0, f_1) *are such that* $f_{M_2} = 0$*, then, for every* $T > 0$,

$$Z_3(1, Tf, u) = Z_3(1, Tf, \bar{u}). \tag{9.10}$$

This is for example the case for the vector fields on $\mathbb{R}^2$ *of* (1.13) *given by* $f_0(x) = (0, x_1^2)$ *and* $f_1(x) = (1, 0)$*, which satisfy* $f_{M_2} = 0$. *For these specific vector fields, (9.9) is even an exact splitting method (that is, of infinite order).*

Of course, due do Theorem 1.13, one could not also have $\zeta_{M_2}(1,u) = \zeta_{M_2}(1,\bar{u})$. *And, indeed, for this control,* $\int_0^1 \int_0^t U = \frac{3}{16} \neq \frac{1}{6}$.

9.2 The second bad Lie bracket

We prove Theorem 1.17, using the coercivity argument of Proposition 7.2.

Proof of Theorem 1.17. We recall from (2.15) that

$$\mathcal{B}_\star^5 = \{X_0\} \cup \{M_i; i \leq 4\} \cup \{W_1, (W_1, X_0), ((W_1, X_0), X_0), W_2, P_{1,1}, (P_{1,1}, X_0), P_{1,2}, Q_1\}, \quad (9.11)$$

using the notations of Definition 1.2, (2.16), as well as $P_{1,1} := (X_1, W_1)$ and $P_{1,2} := (M_1, W_1)$.

Let f_0, f_1 be smooth vector fields on $\mathbb{R}^d$ such that $f_{W_1} = 0$. If $f_{M_\nu} = 0$ for some $\nu \in \{0, 1, 2, 3\}$, then $f_{W_2} = 0$ and the conclusion holds. Indeed, since $W_2 = (M_1, M_2)$, f_{W_2} clearly vanishes if $f_{M_1} = 0$ or $f_{M_2} = 0$. It also obviously vanishes if $f_{M_0} = f_{X_1} = f_1 = 0$ (since then all brackets except f_0 vanish). Eventually, using the Jacobi identity (2.2), one obtains

$$W_2 = [M_1, M_2] = [[W_1, X_0], X_0] - [X_1, M_3]. \quad (9.12)$$

Hence, if $f_{W_1} = 0$ and $f_{M_3} = 0$, then $f_{W_2} = 0$. We can thus assume that $f_{M_\nu} \neq 0$ for all $\nu \in \{0, 1, 2, 3\}$.

We assume that there exists an $(\mathbb{R}^+, \mathbb{R})$ splitting method of order 5 relative to (f_0, f_1). Let $u \in \mathcal{U}$ be the associated control. To lighten the notations, since u is fixed, we write ζ_b instead of $\zeta_b(1, u)$.

The assumption $f_{W_1} = 0$ implies $f_b = 0$ for every

$$b \in \{W_1, (W_1, X_0), ((W_1, X_0), X_0), P_{1,1}, (P_{1,1}, X_0), P_{1,2}, Q_1\}. \quad (9.13)$$

Thus

$$Z_5(1, Tf, u) = T(f_0 + \zeta_{X_1} f_1) + \sum_{\nu=1}^{3} T^{\nu+1} \zeta_{M_\nu} f_{M_\nu} + T^5 \left(\zeta_{M_4} f_{M_4} + \zeta_{W_2} f_{W_2}\right).$$

The Magnus estimate gives

$$e^{Z_5(1,Tf,u)} - e^{T(f_0+f_1)} = \underset{T\to 0}{O}(T^6).$$

By Lemma 9.1 and the fact that $f_1 \neq 0, \dots, f_{M_3} \neq 0$, we obtain $\zeta_{X_1} = 1$ and $\zeta_{M_\nu} = 0$ for $\nu \in \{1, 2, 3\}$. Moreover, we obtain

$$\zeta_{M_4} f_{M_4} + \zeta_{W_2} f_{W_2} = 0. \quad (9.14)$$

By Proposition 7.2, $\zeta_{W_2} + \zeta_{M_4} > 0$, thus both coefficients cannot be simultaneously null and f_{W_2} and f_{M_4} are linearly dependent. □

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