Intrinsic sampling of stochastic dynamics on Riemannian manifolds

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Joint work with Eugen Bronasco, Baptiste Huguet, and Sébastien Macé

GeoStoch, Eindhoven, 2026

ANR project MaStoC - Manifolds and Stochastic Computations

Contents

References of this talk:

• E. Bronasco, A. BL, Hopf algebra structures for the backward error analysis of ergodic stochastic differential equations, *Numer. Math. (2026)*.

• E. Bronasco, A. BL, B. Huguet, High order integration of stochastic dynamics on Riemannian manifolds with frozen flow methods, *arXiv:2503.21855*.

• A. BL, S. Macé, Post-processed frozen-flow methods for the long time sampling of ergodic dynamics on Riemannian manifolds, *In preparation*.

Contents

Riemannian stochastic dynamics

Consider a Riemannian manifold $(\mathcal{M}, \nabla^{\mathrm{LC}})$. Let $E_1, \dots, E_D$ be a smooth frame:

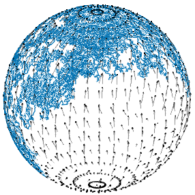
$$\mathrm{Span}_{\mathbb{R}}(E_1(p), \dots, E_D(p)) = T_p\mathcal{M}, \quad y \in \mathcal{M}.$$

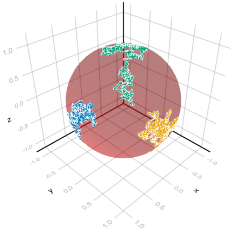
Given a vector field $F(x) = \sum_{d=1}^{D} f^d(x) E_d(x)$, we consider

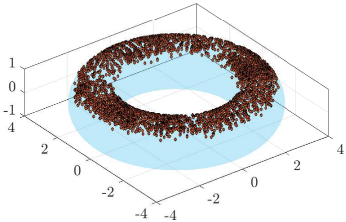
$$dX(t) = F(X(t))dt + \sqrt{2} \sum_{d=1}^{D} E_d(X(t)) \circ dW_d(t).$$

Riemannian Langevin dynamics for E_d orthonormal basis:

$$dX(t) = -\sum_{d=1}^{D} (E_d[V]E_d + \nabla^{\mathrm{LC}}_{E_d} E_d)(X(t))dt + \sqrt{2} \sum_{d=1}^{D} E_d(X(t)) \circ dW_d(t)$$







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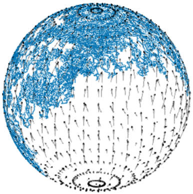
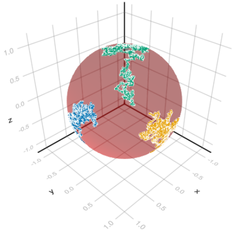
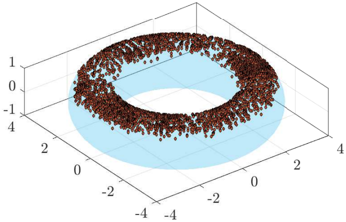






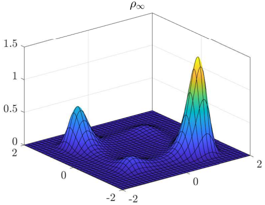
Ergodicity and applications

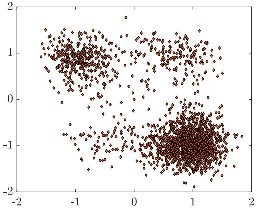
Weak approximations: Given a test function $\phi \in \mathcal{C}_P^\infty(\mathcal{M})$, an integrator is of weak order p if

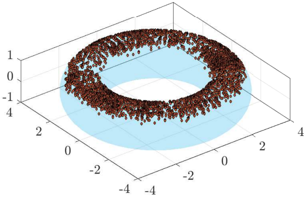
$$|\mathbb{E}[\phi(X_n)] - \mathbb{E}[\phi(X(t_n))]| \leqslant Ch^p, \quad n = 0, \dots, N.$$

Ergodicity property:

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T \phi(X(s))ds = \int_{\mathcal{M}} \phi(y)\rho_\infty(y)d\,\mathrm{vol}(y) \quad \text{almost surely}, \quad \rho_\infty \propto e^{-V}.$$







Applications of sampling on manifolds: geometric statistics, molecular dynamics, stochastic optimisation, ...

The idea of geometric integration

Figure: Numerical simulations of a Brownian motion on the sphere.

The idea of geometric integration

Figure: Numerical simulations of a Brownian motion on the sphere.

Idea: **numerical methods should try to preserve the geometry as much as possible**.

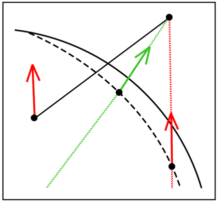
Challenge: a geometry is not "just a manifold". The numerical approaches have to satisfy that **their definition, convergence analysis, and implementation all rely on the same geometric framework as the model.**

Existing stochastic integrators on manifolds

Motivations for finding intrinsic methods:

• Projection methods rely on an embedding in a bigger space, are expensive and unstable.

• The high order theory of projection methods is difficult (BL, 2021 - 7 pages of calculations, **no straightforward algebraic structure**, ∼ 25 order conditions for order 2).



Example (Euler projection integrator)

The most popular integrator is the Euler scheme with explicit projection direction[a]

$$X_{n+1} = X_n + hf(X_n) + \sqrt{2h}\xi_n + \lambda\nabla\zeta(X_n),\ \zeta(X_{n+1}) = 0.$$

[a] Ciccotti, Kapral, Vanden-Eijnden, 2005; Lelièvre, Le Bris, Vanden-Eijnden, 2008; Lelièvre, Rousset, Stolz, 2010; . . .

Example (From Bharath, Lewis, Sharma, Tretyakov, 2024)

The Riemannian Langevin method has order one:

$$X_{n+1} = \exp^{\mathrm{Riem}}_{X_n}(hf(X_n) + \sqrt{2h}g^{-1/2}(X_n)\xi_n)$$

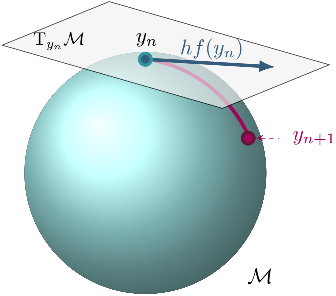
Contents

Connection and geodesic exponential

An affine **connection** $\triangleright: \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ encodes the geometric structure we equip the manifold with.
A **geodesic** $\gamma(t) = \exp(tv)p$ is a curve on $\mathcal{M}$ satisfying

$$\gamma'(t) \triangleright \gamma'(t) = 0, \quad \gamma(0) = p \in \mathcal{M}, \quad \gamma'(0) = v \in T_p\mathcal{M}.$$

Example: the geodesic Euler method for $y' = F(y)$ is $y_{n+1} = \exp(hF(y_n))y_n$.



Example

Euclidean case:

$$g \triangleright f(p) = f'(p)g(p),$$

$$\exp(tv)p = p + tv.$$

Matrix Lie group:

$$\exp(tv)p = \mathrm{Exp}(tv)p.$$

Frame and connection

Let $E_1, \dots, E_D$ be a frame basis (for simplicity):

$$\mathrm{Span}_{\mathbb{R}}(E_1(p), \dots, E_D(p)) = T_p\mathcal{M}, \quad p \in \mathcal{M}.$$

Define the Weitzenböck connection

$$G \rhd F = \sum_{d=1}^{D} G[f^d]E_d, \quad F = \sum_{d=1}^{D} f^d E_d,$$

and the bracket

$$[F, G] = [\![F, G]\!]_J - F \rhd G + G \rhd F = -T(F, G),$$

where $[\![F, G]\!]_J$ is the Jacobi bracket.

Proposition (Ebrahimi-Fard, Lundervold, Munthe-Kaas, '12)

If the frame spans a Lie algebra, the space $(\mathfrak{X}(\mathcal{M}), [-,-], \rhd)$ is a ***post-Lie algebra****:*

$$F \rhd [G, H] = [F \rhd G, H] + [G, F \rhd H],$$
$$[F, G] \rhd H = F \rhd (G \rhd H) - (F \rhd G) \rhd H - G \rhd (F \rhd H) + (G \rhd F) \rhd H.$$

In particular, $\rhd$ has ***constant torsion and vanishing curvature****.*

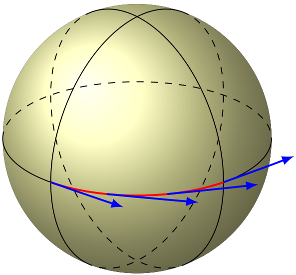
Frozen flows

A **frozen vector field** is

$$F_x(p) = \sum_{d=1}^{D} f^d(x) E_d(p).$$

The **frozen-flow** $\exp(tF_x)p$ is the solution of

$$y'(t) = F_x(y(t)), \quad y(0) = p.$$



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Proposition

The frozen-flow Euler method for

$$dX = F(X)dt + \sqrt{2} E_d(X) \circ dW_d$$

is of weak order one and is given by

$$Y_{n+1} = \exp\left(\sum_{d=1}^{D} \left(hf^d(Y_n) + \sqrt{2h}\xi_n^d \right) E_d \right) Y_n, \quad \xi_n \sim \mathcal{N}(0, I_D)$$

Frozen-flow methods

New frozen-flow integrators[1]

$$H_n^i = \exp\Big(\sum_{d=1}^{D} \Big(h \sum_{j=1}^{s} Z_{i,j,K}^0 f^d(H_n^j) + \sqrt{h} Z_{i,K}^d \Big) E_d \Big) \dots$$

$$\dots \exp\Big(\sum_{d=1}^{D} \Big(h \sum_{j=1}^{s} Z_{i,j,1}^0 f^d(H_n^j) + \sqrt{h} Z_{i,1}^d \Big) E_d \Big) Y_n,$$

$$Y_{n+1} = \exp\Big(\sum_{d=1}^{D} \Big(h \sum_{i=1}^{s} z_{i,K}^0 f^d(H_n^i) + \sqrt{h} z_K^d \Big) E_d \Big) \dots$$

$$\dots \exp\Big(\sum_{d=1}^{D} \Big(h \sum_{i=1}^{s} z_{i,1}^0 f^d(H_n^i) + \sqrt{h} z_1^d \Big) E_d \Big) Y_n.$$

where the coefficients are Gaussian.

Remark: the frozen-flow methods work on ANY smooth Riemannian manifold.

[1] in the spirit of Crouch-Grossman and commutator-free Lie group methods, see Celledoni, Marthinsen, Owren and also Iserles, Munthe-Kaas, Quispel, Zanna, $\sim$ 1990's-2006

Contents

Convergence theorem for weak error

Theorem (Bronasco, BL, Huguet)

Consider a vector field F and a frame E_d that are Lipschitz continuous, $2p+2$-times continuously differentiable, satisfy technical polynomial growth estimates for their derivatives[a]. Denote the Taylor-Talay-Tubaro expansions

$$\mathbb{E}[\phi(X_1)|X_0=x] = \phi(x) + \sum_{j=1}^{p} h^j \mathcal{A}_j \phi(x) + h^{p+1} R_p^h(\phi,x),$$

$$\mathbb{E}[\phi(X(h))|X_0=x] = \phi(x) + \sum_{j=1}^{p} \frac{h^j}{j!} \mathcal{L}^j \phi(x) + h^{p+1} R_p^h(\phi,x).$$

Then, if the operators satisfy

$$\mathcal{A}_j = \frac{1}{j!}\mathcal{L}^j, \quad j=1,\dots p, \quad \mathcal{L}\phi = F[\phi] + \sum_{d=1}^{D} E_d[E_d[\phi]],$$

then the integrator has global weak order p.

[a] in the spirit of the Bakry-Émery criterion $\mathrm{Ric} + \mathrm{Hess}(V) \geqslant \kappa$.

New second weak order intrinsic method

Equation: $dX(t) = F(X(t))dt + \sqrt{2}\sum_{d=1}^{D} E_d(X(t)) \circ dW_d(t)$, $F = f^d E_d$

New explicit frozen flow integrator of weak order two:

$$H_n = \exp\Big(\sum_{d=1}^{D}\Big(\frac{1}{2}hf^d(Y_n) + \sqrt{h}\xi_n^{d,1}\Big)E_d\Big)Y_n$$

$$Y_{n+1} = \exp\Big(\sum_{d=1}^{D}\Big((\frac{\sqrt{2}}{2}-1\big)hf^d(Y_n) + (2-\sqrt{2})hf^d(H_n)$$

$$+ \big(1-\sqrt{2}\big)\sqrt{h}\xi_n^{d,1} + \sqrt{h}\xi_n^{d,2}\Big)E_d\Big)$$

$$\exp\Big(\sum_{d=1}^{D}\Big(\big(1-\frac{\sqrt{2}}{2}\big)hf^d(Y_n) + (\sqrt{2}-1)hf^d(H_n) + \sqrt{2h}\xi_n^{d,1}\Big)E_d\Big)Y_n.$$

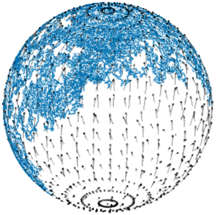
Notes on the implementation:

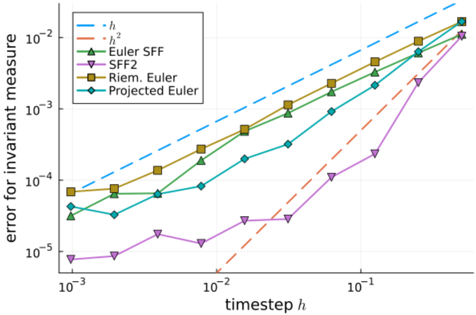
- On homogeneous spaces, exp is the matrix exponential.
- The frozen-flow exponential can be replaced by high-order retractions.
- The geometric operations are already implemented in a handful of packages (see, for instance, Manifolds.jl)

Numerical experiment on the sphere

We now have **3 numerical approaches**:

- Projection: $X_{n+1} = X_n + hF(X_n) + \sqrt{2h}\xi_n + \lambda g(X_n), \quad \zeta(X_{n+1}) = 0,$
- Riemannian: $X_{n+1} = \exp^{\text{Riem}}_{X_n}(hF(X_n) + \sqrt{2h}\xi_n),$
- Frozen-flow : $X_{n+1} = \exp(hF(X_n) + \sqrt{2h}\xi_n^d E_d)X_n.$





The new second order methods outperforms the other integrators in accuracy for a similar cost. It is the **first high-order intrinsic integrator of the literature**.

Convergence theorem for the invariant measure[2]

Theorem (BL, Macé)

Let the Taylor-Talay-Tubaro expansion $\mathbb{E}[\phi(X_1)] = \phi + h\mathcal{A}_1\phi + h^2\mathcal{A}_2\phi + \dots$
Under technical assumptions, if the operators satisfy

$$\mathcal{A}_1 = \mathcal{L}, \quad \mathcal{A}_j^* \rho_\infty d\,\mathrm{vol} = 0, \quad j = 2, \dots p,$$

then the integrator has order p for the invariant measure.

Frozen-flow generalisation of the Leimkuhler-Matthews method:

$$H_n = \exp\left(\left(\frac{\sqrt{2h}}{2}\xi_n^d\right)E_d\right)X_n$$

$$X_{n+1} = \exp\left(\left(\frac{5h}{4}f^d(H_n) + \frac{\sqrt{2h}}{4}\xi_n^d\right)E_d\right)\exp\left(\left(-\frac{h}{4}f^d(H_n) + \frac{3\sqrt{2h}}{4}\xi_n^d\right)E_d\right)X_n$$

$$\overline{X_N} = \exp\left(\frac{\sqrt{2h}}{2}\xi_N^d E_d\right)X_N$$

[2] see also Debussche, Faou, 2012 and Abdulle, Vilmart, Zygalakis, 2014.

Numerical experiment on SO

Sampling of Riemannian Langevin dynamics with a quadratic potential:

$$dX(t) = -\nabla_{\mathcal{M}} V(X(t))dt + \sqrt{2}dW_{\mathcal{M}}(t).$$

The frame simply is

$$E_d(y) = A_d y.$$

Method 1 *

Intrinsic Leimkuhler-Matthews method on SO

$$H_n = \operatorname{Exp}\left(\sqrt{\tfrac{h}{2}}\xi_n^d A_d\right) X_n,$$

$$X_{n+1} = \operatorname{Exp}\left(\left(\tfrac{5h}{4} f^d(H_n) + \tfrac{\sqrt{2h}}{4}\xi_n^d\right) A_d\right) \operatorname{Exp}\left(\left(-\tfrac{h}{4} f^d(H_n) + \tfrac{3\sqrt{2h}}{4}\xi_n^d\right) A_d\right) X_n,$$

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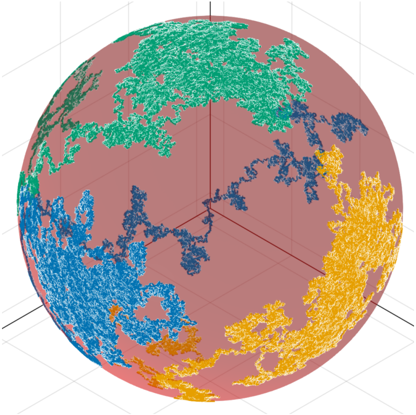
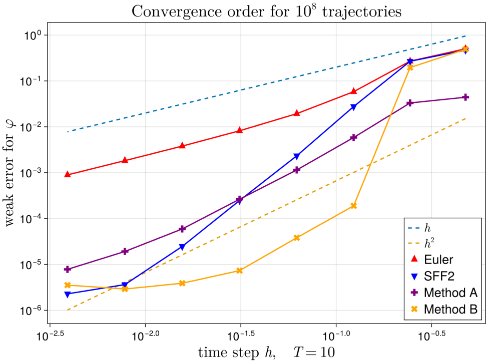


Trajectory on SO_3 with quadratic potential.



Order for the invariant measure for a quadratic potential on SO_3.

Numerical experiment on $\mathbb{S}^2$

Images/Seb_traj_Sphere.png

Sampling the Von-Mises-Fisher distribution.

Images/Seb_error_sphere_10^8.png

Order for the invariant measure for the potential $(x, y, z) \mapsto -25z$.

Contents

Tensor algebra of vector fields

Let the frozen composition in $T(\mathfrak{X}(\mathcal{M}))$ be the differential operator

$$(G \cdot F) \rhd \phi = \sum_{i,j} g^j f^i E_j[E_i[\phi]].$$

Similarly, define the Grossman-Larson product (extend by Guin-Oudom)

$$(G * F) \rhd \phi = \sum_{i,j} g^j E_j[f^i E_i[\phi]] = G \rhd (F \rhd \phi) = (G \rhd F) \rhd \phi + (G \cdot F) \rhd \phi.$$

In $\mathbb{R}^d$, we have

$$(G \cdot F) \rhd \phi = \phi''(G, F), \quad (G * F) \rhd \phi = (\phi' F)' G.$$

Then, $(T(\mathfrak{X}(\mathcal{M})), \cdot, \Delta_{\sqcup\sqcup}, \rhd)$ and $(T(\mathfrak{X}(\mathcal{M})), *, \Delta_{\sqcup\sqcup})$ are (post-)Hopf algebras.

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Proposition

The Taylor expansions of the geodesic and exact flow exponentials are

$$\phi(\exp(F_p)p) = \exp^{\cdot}(F) \rhd \phi(p), \quad \phi(\exp(F)p) = \exp^*(F) \rhd \phi(p),$$

where $\exp^*(F) = \mathrm{id} + F + \frac{1}{2!} F * F + \frac{1}{3!} F * F * F + \dots$.

Example of the Euler method

The Taylor expansion of the Euler frozen-flow method is

$$\phi(\exp(hF_p+\sqrt{2h}\xi^d E_d)p)=\exp^{\cdot}(hF+\sqrt{2h}\xi^d E_d)\rhd\phi$$

[3] see Isserlis-Wick theorem.

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The Taylor expansion of the Euler frozen-flow method is

$$\begin{aligned}\phi(\exp(hF_p+\sqrt{2h}\xi^d E_d)p) &= \exp^{\cdot}(hF+\sqrt{2h}\xi^d E_d)\rhd\phi\\ &= \Big(\mathrm{id}+h^{1/2}\sqrt{2}\xi^d E_d + h(F+\xi^{d_2}\xi^{d_1}E_{d_2}\cdot E_{d_1})\\ &+ h^{3/2}(\frac{\sqrt{2}}{2}\xi^d F\cdot E_d+\frac{\sqrt{2}}{2}\xi^d E_d\cdot F+\frac{\sqrt{2}}{3}\xi^{d_3}\xi^{d_2}\xi^{d_1}E_{d_3}\cdot E_{d_2}\cdot E_{d_1})+\dots\Big)\rhd\phi.\end{aligned}$$

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$$\phi(\exp(hF_p+\sqrt{2h}\xi^d E_d)p)=\exp^{\cdot}(hF+\sqrt{2h}\xi^d E_d)\rhd\phi$$
$$=\Big(\mathrm{id}+h^{1/2}\sqrt{2}\xi^d E_d+h(F+\xi^{d_2}\xi^{d_1}E_{d_2}\cdot E_{d_1})$$
$$+h^{3/2}(\frac{\sqrt{2}}{2}\xi^d F\cdot E_d+\frac{\sqrt{2}}{2}\xi^d E_d\cdot F+\frac{\sqrt{2}}{3}\xi^{d_3}\xi^{d_2}\xi^{d_1}E_{d_3}\cdot E_{d_2}\cdot E_{d_1})+\dots\Big)\rhd\phi.$$

Then, the expectation pairs the Gaussians together[3] and yields

$$\mathbb{E}[\phi(\exp(hF_p+\sqrt{2h}\xi^d E_d)p)]=\Big(\mathrm{id}+h(F+E_d\cdot E_d)$$
$$+h^2(\frac{1}{2}F\cdot F+\frac{1}{3}F\cdot E_d\cdot E_d+\frac{1}{3}E_d\cdot F\cdot E_d+\frac{1}{3}E_d\cdot E_d\cdot F$$
$$+\frac{1}{6}E_{d_2}\cdot E_{d_2}\cdot E_{d_1}\cdot E_{d_1}+\frac{1}{6}E_{d_2}\cdot E_{d_1}\cdot E_{d_2}\cdot E_{d_1}+\frac{1}{6}E_{d_1}\cdot E_{d_2}\cdot E_{d_2}\cdot E_{d_1})$$
$$+\dots\Big)\rhd\phi.$$

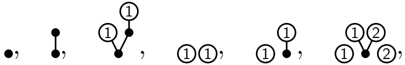
[3] see Isserlis-Wick theorem.

Planar exotic forests[4]

Definition

A planar exotic forest is an ordered list of planar trees decorated by $\mathbb{N}$ s.t.
- $\bullet$ stands for the decoration 0,
- the other decorations only appear 0 or 2 times, and only on leaves.

Examples of exotic planar forests:

$$\mathcal{EF} = \mathrm{Span}_{\mathbb{R}}(\mathbf{1},$$

$$\ldots)$$

[4]see L., Vilmart, 2020-2022; L., Munthe-Kaas, 2024

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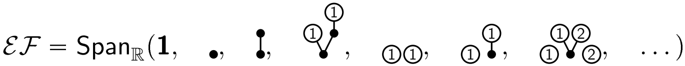

Difficulty: An exotic forest is NOT a concatenation of exotic trees in general:

$$①① \neq ① \cdot ①.$$

[4] see L., Vilmart, 2020-2022; L., Munthe-Kaas, 2024

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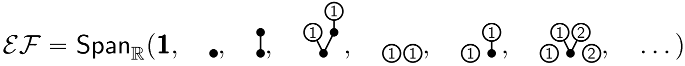
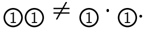
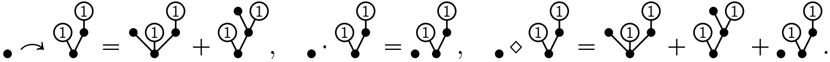


Difficulty: An exotic forest is NOT a concatenation of exotic trees in general:



$\mathcal{EF}$ is equipped with grafting $\curvearrowright$, concatenation $\cdot$, and Grossman-Larson $\diamond$:



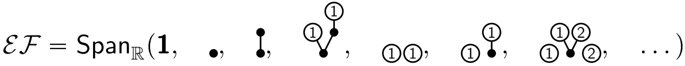
[4] see L., Vilmart, 2020-2022; L., Munthe-Kaas, 2024

Planar exotic forests[4]

Definition

A planar exotic forest is an ordered list of planar trees decorated by $\mathbb{N}$ s.t.
- $\bullet$ stands for the decoration 0,
- the other decorations only appear 0 or 2 times, and only on leaves.

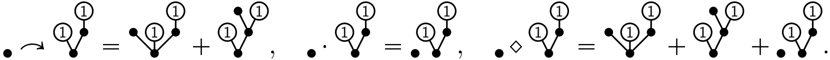
Examples of exotic planar forests:



Difficulty: An exotic forest is NOT a concatenation of exotic trees in general:

$$①① \neq ① \cdot ①.$$

$\mathcal{EF}$ is equipped with grafting $\curvearrowright$, concatenation $\cdot$, and Grossman-Larson $\diamond$:



Theorem

$(\mathcal{EF}, \cdot, \Delta_{\sqcup\!\sqcup})$ and $(\mathcal{EF}, \diamond, \Delta_{\sqcup\!\sqcup})$ are Hopf algebras.

[4] see L., Vilmart, 2020-2022; L., Munthe-Kaas, 2024

Elementary differential map

The elementary differential map $\mathbb{F}\colon \mathcal{EF} \to T(\mathfrak{X}(\mathcal{M}))$ translates from exotic forests to differential operators:

$$\mathbb{F}(\bullet) = F, \quad \mathbb{F}(\substack{\bullet\\ |\\ \bullet}) = F \rhd F, \quad \mathbb{F}({}_{\textcircled{1}}\substack{\textcircled{1}\\ |\\ \bullet}) = E_d \cdot (E_d \rhd F), \quad \mathbb{F}(\substack{\textcircled{1}\\ |\\ \bullet}{}_{\textcircled{1}}) = (E_d \rhd F) \cdot E_d.$$

Proposition

$\mathbb{F}$ *is a morphism:* $\mathbb{F}(\pi_1 \curvearrowright \pi_2) = \mathbb{F}(\pi_1) \rhd \mathbb{F}(\pi_2)$,

$$\mathbb{F}(\pi_1 \cdot \pi_2) = \mathbb{F}(\pi_1) \cdot \mathbb{F}(\pi_2), \quad \mathbb{F}(\pi_1 \diamond \pi_2) = \mathbb{F}(\pi_1) * \mathbb{F}(\pi_2).$$

Example (frozen-flow Euler method)

$$\begin{aligned}
\mathbb{E}[\phi(\exp(hF_p + \sqrt{2h}\xi^d E_d)p)] &= \mathbb{F}\Big(\mathbf{1} + h(\bullet + \textcircled{1}\textcircled{1}) + h^2(\frac{1}{2}\bullet\ \bullet + \frac{1}{3}\bullet\ \textcircled{1}\textcircled{1} \\
&+ \frac{1}{3}\textcircled{1}\bullet\textcircled{1} + \frac{1}{3}\textcircled{1}\textcircled{1}\bullet + \frac{1}{6}\textcircled{2}\textcircled{2}\textcircled{1}\textcircled{1} + \frac{1}{6}\textcircled{2}\textcircled{1}\textcircled{2}\textcircled{1} + \frac{1}{6}\textcircled{1}\textcircled{2}\textcircled{2}\textcircled{1}) + \dots\Big) \rhd \phi(p) \\
&= \Big(\mathrm{id} + h(F + E_d \cdot E_d) + h^2(\frac{1}{2}F \cdot F + \frac{1}{3}F \cdot E_d \cdot E_d + \frac{1}{3}E_d \cdot F \cdot E_d \\
&+ \frac{1}{3}E_d \cdot E_d \cdot F + \frac{1}{6}E_{d_2} \cdot E_{d_2} \cdot E_{d_1} \cdot E_{d_1} + \frac{1}{6}E_{d_2} \cdot E_{d_1} \cdot E_{d_2} \cdot E_{d_1} + \dots\Big) \rhd \phi(p).
\end{aligned}$$

Weak order conditions

Exotic forest π	Differential $\mathbb{F}(\pi)[\phi]$	Order condition $a(\pi) = e(\pi)$
•	$f^i E_i[\phi]$	$z^0_{i,k_1} = 1$
①①	$E_{d_1}[E_{d_1}[\phi]]$	$\sum^!_{k_1 \geqslant k_2} \mathbb{E}[z^{d_1}_{k_2} z^{d_1}_{k_1}] = 1$
•	$f^j E_j[f^i]E_i[\phi]$	$z^0_{i,k_2} Z^0_{i,j,k_1} = \frac{1}{2}$
①①	$E_{d_1}[E_{d_1}[f^i]]E_i[\phi]$	$\sum^!_{k_2 \geqslant k_3} \mathbb{E}[Z^{d_1}_{i,k_3} Z^{d_1}_{i,k_2}] z^0_{i,k_1} = \frac{1}{2}$
	$f^j f^i E_j[E_i[\phi]]$	$\sum^!_{k_1 \geqslant k_2} z^0_{j,k_2} z^0_{i,k_1} = \frac{1}{2}$
	$E_{d_1}[f^i]E_i[E_{d_1}[\phi]]$	$\sum^!_{k_1 \geqslant k_2} z^0_{i,k_2} \mathbb{E}[Z^{d_1}_{i,k_3} z^{d_1}_{k_1}] = 0$
• •	$E_{d_1}[f^i]E_{d_1}[E_i[\phi]]$	$\sum^!_{k_2 \geqslant k_1} z^0_{i,k_2} \mathbb{E}[Z^{d_1}_{i,k_3} z^{d_1}_{k_1}] = 1$
	$f^i E_i[E_{d_1}[E_{d_1}[\phi]]]$	$\sum^!_{k_1 \geqslant k_2 \geqslant k_3} z^0_{i,k_3} \mathbb{E}[z^{d_1}_{k_2} z^{d_1}_{k_1}] = \frac{1}{2}$
	$f^i E_{d_1}[E_i[E_{d_1}[\phi]]]$	$\sum^!_{k_1 \geqslant k_3 \geqslant k_2} z^0_{i,k_3} \mathbb{E}[z^{d_1}_{k_2} z^{d_1}_{k_1}] = 0$
• ①①	$f^i E_{d_1}[E_{d_1}[E_i[\phi]]]$	$\sum^!_{k_3 \geqslant k_1 \geqslant k_2} z^0_{i,k_3} \mathbb{E}[z^{d_1}_{k_2} z^{d_1}_{k_1}] = \frac{1}{2}$
① • ①	$E_{d_2}[E_{d_2}[E_{d_1}[E_{d_1}[\phi]]]]$	$\sum^!_{k_1 \geqslant k_2 \geqslant k_3 \geqslant k_4} \mathbb{E}[z^{d_2}_{k_4} z^{d_2}_{k_3}] \mathbb{E}[z^{d_1}_{k_2} z^{d_1}_{k_1}] = \frac{1}{2}$
①① •	$E_{d_2}[E_{d_1}[E_{d_2}[E_{d_1}[\phi]]]]$	$\sum^!_{k_1 \geqslant k_2 \geqslant k_3 \geqslant k_4} \mathbb{E}[z^{d_2}_{k_4} z^{d_2}_{k_2}] \mathbb{E}[z^{d_1}_{k_3} z^{d_1}_{k_1}] = 0$
②②①①	$E_{d_1}[E_{d_2}[E_{d_2}[E_{d_1}[\phi]]]]$	$\sum^!_{k_1 \geqslant k_2 \geqslant k_3 \geqslant k_4} \mathbb{E}[z^{d_2}_{k_3} z^{d_2}_{k_2}] \mathbb{E}[z^{d_1}_{k_4} z^{d_1}_{k_1}] = 0$

Primitive elements and shuffle relations

Shuffle product:

$$\text{②①②①} \sqcup\!\sqcup \text{①②②①} = \bullet + \text{①①} + \bullet\,\text{①①},$$
$$\text{①}\bullet\text{①} \sqcup\!\sqcup \text{①①}\,\bullet = 2\,\text{②②} + 2\,\text{①①} + 2\,\text{②②①①},$$

The coefficient maps of numerical flows are characters of $(\mathcal{EF}, \sqcup\!\sqcup)$:

$$a(\pi_1 \sqcup\!\sqcup \pi_2) = a(\pi_1)a(\pi_2).$$

Following Owren, '06, we have shuffle relations:

$$a(\text{②①②①})^2 = 2a(\text{②①①②}),$$
$$a(\bullet)a(\bullet\ \bullet) = a(\bullet) + a(\text{①①}) + a(\bullet\,\text{①①}),$$
$$a(\text{①①}\,\bullet)^2 = 2a(\text{①}\bullet\text{①}) + 2a(\text{①①}) + 2a(\text{②②①①}).$$

Proposition

The order conditions are indexed by the exotic forests, modulo the shuffle relations. In particular, there are 2, 8, and 73 conditions for order 1,2, and 3 (against 2, 11, and 95 exotic forests).

Conclusion

Summary:

- We provide a brand new class of intrinsic high-order methods for solving stochastic dynamics on manifolds in the weak sense and for the invariant measure.
- We give a Riemannian convergence analysis in both cases.
- The order theory relies on a new formalism of planar exotic forests, which extends the existing deterministic works.
- Exotic forests now appear in quantum physics, in the approximation of PDEs, the study of turbulence,. . .

Outlooks:

- The analysis and implementation rely on an artificial connection. Ongoing extension of Lie-group methods to any geometry (with S. Carrier, E. Grong, H. Munthe-Kaas).
- Implementation of the new methods in the Julia package Manifolds.jl (with R. Bergmann and S. Macé).
- Characterisation of exact sampling methods with the exotic aromatic bicomplex (with V. Dotsenko, P. Laubie, J. Sao Joao).
- **Algebraic and geometric foundations of intrinsic geometric integration** (ANR MaStoC team: S. Carrier, S. Macé, S.G. Venkatesh,...)

Exotic MKW structure

Theorem

Let the Munthe-Kaas-Wright coproduct:

$$\Delta_{MKW}(\tau) := \sum_{\text{adm. cut } c} P^c(\tau) \otimes R^c(\tau), \quad \Delta_{MKW}(\pi) \quad := (id \otimes B^-)\Delta_{MKW}(B^+(\pi)).$$

Then, $(\mathcal{EF}, \sqcup\!\sqcup, \Delta_{MKW})$ *is a Hopf algebra dual to* $(\mathcal{EF}, \diamond, \Delta_{\sqcup\!\sqcup})$*. Its convolution product represents the composition of exotic series:*

$$S(a) \circ S(b) = S(a * b), \quad a * b = \mu_{\mathbb{R}} \circ (a \otimes b) \circ \Delta_{MKW},$$

Example